Syllabus Lie Groups, General Theory

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Part of these notes closely follow Chapters 1, 2 and 5 in G. Segal's lecture notes on Lie groups, which form the middle part of the book:

R. Carter, G. Segal and I. Macdonald, *Lectures on Lie groups and Lie algebras*, London Mathematical Society Student Texts 32, Cambridge University Press, 1995.

Chapter 1. Generalities about topological groups

1.1 Definition A *topological group* is a set G which is both a group and a topological space such that the group operations $(x,y) \mapsto xy\colon G \times G \to G$ and $x \mapsto x^{-1}\colon G \to G$ are continuous.
Any subgroup of a topological group G becomes a topological group in the relative topology of G.
A map $\phi\colon G \to H$ from a topological group G to a topological group H is called a *homomorphism of topological groups* if ϕ is both a group homomorphism and a continuous map.
Two topological groups G and H are called *isomorphic* if there is an invertible map $\phi\colon G \to H$ such that both ϕ and ϕ^{-1} are homomorphisms of topological groups.

1.2 Examples of topological groups

(a) Any group with the discrete topology (a so-called *discrete topological group*). In particular, any finite group will be silently assumed to be a topological group in this way.

(b) $\mathbb{R}^n$ as additive group with the topology of $\mathbb{R}^n$. Similarly $\mathbb{C}^n$ as additive group with the topology of $\mathbb{C}^n$. Note that $\mathbb{C}^n$ is isomorphic to $\mathbb{R}^{2n}$ as a topological group.

(c) $GL(n,\mathbb{R})$, the group of $n \times n$ real invertible matrices with topology as subset of $\mathbb{R}^{n^2}$ (by considering the matrix elements T_{ij} of $T \in GL(n,\mathbb{R})$ as coordinates).

(d) $GL(n,\mathbb{C})$, the group of $n \times n$ complex invertible matrices with topology as subset of $\mathbb{C}^{n^2}$ (by considering the matrix elements T_{ij} of $T \in GL(n,\mathbb{C})$ as coordinates).

(e) Any subgroup of $GL(n,\mathbb{R})$ or of $GL(n,\mathbb{C})$ in the relative topology. Among such subgroups the *closed* subgroups (subgroups G which are closed subsets of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$) have much nicer properties than the non-closed subgroups, and we will mostly restrict attention to these closed subgroups.

(f) The groups $SL(n,\mathbb{R}) := \{T \in GL(n,\mathbb{R}) \mid \det T = 1\}$ and $SL(n,\mathbb{C}) := \{T \in GL(n,\mathbb{C}) \mid \det T = 1\}$.

(g) $O(n)$, the group of real orthogonal $n \times n$ matrices, and its subgroup $SO(n) := \{T \in O(n) \mid \det T = 1\}$.

(h) $U(n)$ the group of complex unitary $n \times n$ matrices and its subgroup $SU(n) := \{T \in U(n) \mid \det T = 1\}$.

Ex. 1.3 Prove that $GL(n,\mathbb{R})$ is a closed subgroup of $GL(n,\mathbb{C})$ (when embedded in $GL(n,\mathbb{C})$ in the obvious way). Show also that $GL(n,\mathbb{C})$ is isomorphic as topological group to a certain closed subgroup of $GL(2n,\mathbb{R})$. Show next that every closed subgroup of $GL(n,\mathbb{R})$ is isomorphic as topological group to some closed subgroup of $GL(n,\mathbb{C})$ and that every closed subgroup of $GL(n,\mathbb{C})$ is isomorphic as topological group to some closed subgroup of $GL(2n,\mathbb{R})$.

Ex. 1.4 Show for all examples in §1.2 that these are topological groups. Show that the examples (f), (g), (h) are closed subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$.

1.5 Proposition Let G be a topological group. If $\{e\}$ is a closed subset of G then G as a topological space is Hausdorff.

Ex. 1.6 Prove Proposition 1.5.

1.7 Remark An important class of topological groups are the *locally compact groups*: these are topological groups which are locally compact Hausdorff spaces. A subclass if formed by the *compact groups*: these are topological groups which are compact Hausdorff spaces.

Ex. 1.8 Show that the discrete topological groups are locally compact and that the compact groups among these are precisely the finite groups. Show also that $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ and all their closed subgroups are locally compact.

1.9 Proposition Let G be a topological group. Then the connected component G_0 of G containing e is a closed normal subgroup of G. The whole collection of connected components of G is precisely the collection of left cosets of G with respect to G_0.

1.10 Example The group $G := O(n)$ is not connected, since the continuous map $\det\colon G \to \mathbb{R}$ maps G onto the subset $\{-1,1\}$ of $\mathbb{R}$, which is not connected. It can be shown that here $G_0 = SO(n)$. For $n := 2$ this is seen very easily. For general n we prove below that $SO(n)$ is connected.

1.11 Definition Let G be a group and X a set. A *group action* of G on X is a map $(g,x) \mapsto g\cdot x\colon G\times X \to X$ such that
(a) $(gh)\cdot x = g\cdot(h\cdot x)$ for all $g,h \in G$ and $x \in X$;
(b) $e\cdot x = x$ for all $x \in X$.
If G is a topological group and X a topological space and if the above mapping $(g,x) \mapsto g\cdot x\colon G\times X \to X$ is continuous then we call the group action a *topological group action* or a *continuous group action*.

If a group action of G on X satisfies the property that for each $x,y \in X$ there exists $g \in G$ such that $g\cdot x = y$ then we call the group action *transitive*.

If G is a group with subgroup H then we denote by G/H the collection of left cosets gH ($g \in G$). Then G acts on G/H by the rule $g'\cdot(gH) := (g'g)H$ and this action is transitive (check this). Moreover, for $g \in G$ we have: $g\cdot(eH) = eH$ iff $g \in H$. The set G/H with G acting on it is called a *homogeneous space*.

Conversely, if we have a transitive group action of G on X and if we fix $x_0 \in X$ then $H := \{g \in G \mid g\cdot x_0 = x_0\}$ is a subgroup of G (the so-called *stabilizer* of x_0 in G)

and the map $g \cdot x_0 \mapsto gH: X \to G/H$ is a well-defined bijection such that $g' \cdot (g \cdot x_0)$ is mapped by this bijection to $g' \cdot (gH)$ (i.e., the actions of G on X and on G/H correspond with each other under this bijection). If G is a topological Hausdorff group and X is a topological Hausdorff space and if the transitive group action of G on X is continuous then the stabilizer subgroup H of x_0 in G is a closed subgroup.

1.12 Example Some examples of transitive continuous actions of groups are:

(a) The symmetric group S_n acting on the set $\{1, 2, \ldots, n\}$. The stabilizer of 1 in S_n is isomorphic to S_{n-1}.

(b) The group $O(n)$ acting on the unit sphere S^{n-1} in $\mathbb{R}^n$. The stabilizer subgroup of $(1, 0, \ldots, 0)$ is the group $O(n-1)$ which is embedded as a closed subgroup of $O(n)$ by

$$T \mapsto \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & T & \\ 0 & & & \end{pmatrix} : O(n-1) \hookrightarrow O(n).$$

(c) The group $SO(n)$ acting on the unit sphere S^{n-1} in $\mathbb{R}^n$. The stabilizer subgroup of $(1, 0, \ldots, 0)$ is the group $SO(n-1)$ which is embedded as a closed subgroup of $SO(n)$ as in (b).

Ex. 1.13 Prove the statements in the last two paragraphs of Definition 1.11. Prove also that the examples in §1.12 indeed give transitive continuous actions of topological groups and that the stabilizer subgroups are the ones mentioned there.

1.14 Proposition The group $SO(n)$ is connected.

Proof Clearly, $SO(2)$ is connected. The proof will use induction with respect to n. The induction hypothesis will follow by the existence of a continuous surjective map $SO(n-1) \times SO(2) \times SO(n-1) \to SO(n)$. This map is obtained by observing that each $T \in SO(n)$ can be written as

$$T = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & \ldots & 0 \\ \sin\theta & \cos\theta & 0 & \ldots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

for some $A, B \in SO(n-1)$ and $\theta \in [0, \pi)$. We prove this last statement. Let $T \in SO(n)$ and let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Then $Te_1 = \cos\theta\, e_1 + \sin\theta\, \xi$ for some $\theta \in [0, \pi)$ and $\xi = (0, \xi_2, \ldots, \xi_n)$ with $|\xi| = 1$. Then, by transitivity of the action of $SO(n-1)$ on S^{n-2}, there exists $A \in SO(n-1)$ such that

$$Te_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & \ldots & 0 \\ \sin\theta & \cos\theta & 0 & \ldots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix} e_1.$$

Denote the product of the two matrices on the right-hand side of the last identity by T'. Then $T' \in SO(n)$ and $(T')^{-1}Te_1 = e_1$. Hence, since $SO(n-1)$ is the stabilizer of e_1 in $SO(n)$, there exists $B \in SO(n-1)$ such that $(T')^{-1}T = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$. □

1.15 *The group of isometries of* $\mathbb{R}^n$ is denoted by $E(n)$. Then $E(n) = \{\tau_b A \mid b \in \mathbb{R}^n,\ A \in O(n)\}$, where $\tau_b\colon v \mapsto v + b\colon \mathbb{R}^n \to \mathbb{R}^n$ denotes a translation. Note that $\tau_b A = \tau_{b'} A'$ iff $b = b'$ and $A = A'$, so, as sets, we can identify $\tau_b A \leftrightarrow (b, A)\colon E(n) \leftrightarrow \mathbb{R}^n \times O(n)$. Via this bijection we can transplant the group operations

$$(\tau_b A)(\tau_{b'} A') = \tau_{Ab'+b}(AA'), \quad \text{unit } \tau_0 I, \quad (\tau_b A)^{-1} = \tau_{-A^{-1}b} A^{-1}$$

on $E(n)$ to group operations

$$(b, A)(b', A') = (b + Ab', AA'), \quad \text{unit } (0, I), \quad (b, A)^{-1} = (-A^{-1}b, A^{-1})$$

on $\mathbb{R}^n \times O(n)$. Now $\mathbb{R}^n \times O(n)$, equipped with the product topology from the topologies of $\mathbb{R}^n$ and $O(n)$, becomes a topological group (check this). Hence $E(n)$ receives via the bijection a topology by which it becomes a topological group.

1.16 Definition Let H and N be groups, let $\mathrm{Aut}(N)$ be the group of automorphisms of N, and let $\alpha\colon H \to \mathrm{Aut}(N)$ be a group homomorphism. Then the *semidirect product* $H \ltimes N$ is the set $N \times H$ made into a group with group operations

$$(n_1, h_1)(n_2, h_2) = (n_1\, \alpha(h_1) n_2, h_1 h_2), \quad \text{unit } (e, e), \quad (n, h)^{-1} = ((\alpha(h^{-1})n)^{-1}, h^{-1}).$$

Now we have group embeddings

$$\begin{aligned} n &\mapsto (n, e)\colon N \hookrightarrow H \ltimes N \quad \text{as a normal subgroup,} \\ h &\mapsto (e, h)\colon H \hookrightarrow H \ltimes N \quad \text{a s a subgroup.} \end{aligned}$$

Thus we can write n instead of (n, e) and h instead of (e, h), and we can write a general element of $H \ltimes N$ as nh ($n \in N$, $h \in H$), since $nh = (n, e)(e, h) = (n, h)$. Then we have

$$hnh^{-1} = \alpha(h)n \quad \text{as shorthand for} \quad (e, h)(n, e)(e, h)^{-1} = (\alpha(h)n, e) \qquad (h \in H,\ n \in N).$$

If N and H are topological groups and if the map $(n, h) \mapsto \alpha(h)n\colon N \times H \to N$ is continuous then $N \times H$ equipped with the product topology becomes a topological group.

1.17 Definition The *Heisenberg group* is the group N defined as the topological space $\mathbb{R}^3$ embedded as a subgroup of $GL(3, \mathbb{R})$ by

$$(a, b, c) \mapsto \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

The group structure on N as a subgroup of $GL(3, \mathbb{R})$ then defines the following structure of topological group on $\mathbb{R}^3$:

$$(a, b, c)(a', b', c') = (a+a', b+b', c+c'+ab'), \quad \text{unit } (0, 0, 0), \quad (a, b, c)^{-1} = (-a, -b, -c+ab).$$

Then the center of N equals $\{(0, 0, c) \mid c \in \mathbb{R}\}$. Let Z be the discrete subgroup $\{(0, 0, c) \mid c \in \mathbb{Z}\}$ of the center. Then Z is a normal subgroup of N. In some literature the quotient group N/Z is called the Heisenberg group.

Ex. 1.18 For $a, b, c \in \mathbb{R}$ define unitary operators T_a, M_b and U_c acting on the Hilbert space $L^2(\mathbb{R})$ as follows:

$$(T_a f)(x) := f(x - a), \quad (M_b f)(x) := e^{2\pi i b x} f(x), \quad (U_c f)(x) := e^{2\pi i c} f(x).$$

Show that the map $(a, b, c) \mapsto T_a M_b U_{-c+ab}$ defines a group homomorphism from the Heisenberg group N (and also from N/Z) into the group of unitary operators on $L^2(\mathbb{R})$.

Chapter 2. $SU(2)$, $SO(3)$ and $SL(2,\mathbb{R})$

2.1 Definition The space of *quaternions* is a four-dimensional real vector space $\mathbb{H}$ with basis 1, i, j, k. We introduce an associative multiplication on $\mathbb{H}$ by the product rules $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Write a general element q of $\mathbb{H}$ as $q = t + xi + yj + zk$ $(t,x,y,z \in \mathbb{R})$. Define an involution $q \mapsto q^*$ on $\mathbb{H}$ by $(t + xi + yj + zk)^* := t - xi - yj - zk$. Then $q \mapsto q^*$ is linear, $(q^*)^* = q$, $1^* = 1$, and $*$ is anti-multiplicative: $(q_1q_2)^* = q_2^*\, q_1^*$. (We say that $\mathbb{H}$ is a real associative $*$-algebra with identity.) We write $\mathbb{H}$ as the direct sum $\mathbb{H} = \mathbb{R} + \mathbb{R}^3$, where $\mathbb{R}$ is the real span of 1 and $\mathbb{R}^3$ is the real span of i, j, k. In $q = t + xi + yj + zk$ we call $t = \frac{1}{2}(q + q^*) \in \mathbb{R}$ the *real part* of q and $v := xi + yj + zk = \frac{1}{2}(q - q^*) \in \mathbb{R}^3$ the *vector part* of q. Then put $|q|^2 := qq^* = q^*q = t^2 + |v|^2 = t^2 + x^2 + y^2 + z^2$ (the squared norm). We have $|q| = |q^*|$ and $|q_1q_2| = |q_1|\,|q_2|$.

2.2 Proposition We have the following correspondences:

$$\begin{array}{rcll}
\mathbb{H} & \longleftrightarrow & \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C} \right\} & \\
q := t + xi + yj + zk & \longleftrightarrow & Q := \begin{pmatrix} t + ix & y + iz \\ -y + iz & t - ix \end{pmatrix} & (\mathbb{R}\text{-linear and bijective}) \\
q_1q_2 & \longleftrightarrow & Q_1Q_2 & \\
q^* & \longleftrightarrow & Q^* & \\
t & = & \frac{1}{2}\operatorname{tr} Q & \\
|q|^2 & = & \det Q & \\
|q| = 1 & \Longleftrightarrow & Q \in SU(2) &
\end{array}$$

The last correspondence gives a topological group isomorphism between the group of quaternions of unit norm and the group $SU(2)$. Since the unit sphere in $\mathbb{R}^4$ is connected (and simply connected), the group $SU(2)$ is connected (and simply connected).

2.3 Proposition For $v \in \mathbb{R}^3 \subset \mathbb{H}$ and $g \in \mathbb{H}\backslash\{0\}$ put $T_g v := gvg^{-1}$. Then T_g is an orthogonal transformation of $\mathbb{R}^3$. In particular, the map $g \mapsto T_g$ is a continuous homomorphism from $SU(2)$ (identified with the group of quaternions of unit norm) to $SO(3)$.

Proof T_g is an orthogonal transformation of $\mathbb{R}^3$ since $\operatorname{tr}(GVG^{-1}) = \operatorname{tr} V = 0$ and $\det(GVG^{-1}) = \det(V)$. The map $g \mapsto T_g$ is a group homomorphism since $(g_1g_2)v(g_1g_2)^{-1}$ $= g_1(g_2vg_2^{-1})g_1^{-1}$. It maps $SU(2)$ into $SO(3)$ since the continuous image of the connected group $SU(2)$ must lie inside the connected component $SO(3)$ of $O(3)$. □

2.4 Theorem The map $g \mapsto T_g$ is a continuous surjective group homomorphism from $SU(2)$ onto $SO(3)$ with kernel consisting of 2 elements $\pm I$.

Proof Use that

$$(t_1 + v_1)(t_2 + v_2) = \big(t_1t_2 - \langle v_1, v_2\rangle\big) + \big(t_1v_2 + t_2v_1 + v_1 \times v_2\big),$$

where $v_1 \times v_2$ is the vector product in $\mathbb{R}^3$ of vectors v_1 and v_2 in $\mathbb{R}^3$. Any rotation T of $\mathbb{R}^3$ which is not the identity can be described by a unit vector u (yielding the rotation axis) and by its angle of rotation 2θ $(0 < \theta < \pi)$, with rotation taken positively with respect to the direction of u. There are precisely two possible ways of describing T in such a way, namely by $u, 2\theta$ and by $-u, 2(\pi - \theta)$.

On the other hand, any quaternion of unit norm not equal to ± 1 can be uniquely represented as $\cos\theta + u \sin\theta$, with u a unit vector in $\mathbb{R}^3$ and $0 < \theta < \pi$. We show that, starting with this representation, the map $v \mapsto (\cos\theta + u\sin\theta)\, v\, (\cos\theta + u \sin\theta)^{-1}$ is a rotation about u through angle 2θ. Indeed,

$$(\cos\theta + u\sin\theta)\, v\, (\cos\theta - u\sin\theta) = \langle u, v\rangle \sin^2\theta\, u + \cos^2\theta\, v - \sin^2\theta\,(u \times v)\times u + \sin(2\theta)\, u \times v.$$

If $v = u$ then this equals u. If v is a unit vector orthogonal to u then this equals $\cos(2\theta)\, v + \sin(2\theta)\, u \times v$. Thus we get the desired rotation, because in the latter case u, v and $u \times v$ form an orthonormal system of unit vectors in the right orientation. □

Thus the connected and simply connected group $SU(2)$ is the two-sheeted covering of the connected group $SO(3)$. We call $SU(2)$ therefore the *universal covering group* of $SO(3)$.

2.5 Theorem For $g_1, g_2 \in SU(2)$ (identified with the group of quaternions of unit norm) put $T_{g_1,g_2} v := g_1 v g_2^{-1}$ $(v \in \mathbb{H})$. Then the map $(g_1, g_2) \mapsto T_{g_1,g_2}$ is a continuous surjective group homomorphism from $SU(2) \times SU(2)$ onto $SO(4)$ with kernel consisting of 2 elements $\pm(I, I)$.

Proof T_{g_1,g_2} is an orthogonal transformation of $\mathbb{R}^4 \simeq \mathbb{H}$ since $\det(G_1 V G_2^{-1}) = \det V$. The map $(g_1, g_2) \mapsto T_{g_1,g_2}$ is a group homomorphism: $(g_1 g_1')v(g_2 g_2')^{-1} = g_1(g_1' v (g_2')^{-1})g_2^{-1}$. It maps $SU(2) \times SU(2)$ into $SO(4)$ since the continuous image of the connected group $SU(2) \times SU(2)$ must lie inside the connected component $SO(4)$ of $O(4)$.

Next we prove that the homomorphism is surjective. Let $T \in SO(4)$ and put $h := T1$. Then h is a quaternion of unit norm, so $h^{-1}\, T1 = 1$ and the map $v \mapsto h^{-1}\, Tv \colon \mathbb{H} \to \mathbb{H}$ is in $SO(4)$. Hence the map $v \mapsto h^{-1}\, Tv \colon \mathbb{H} \to \mathbb{H}$ is in $SO(3)$ (the subgroup of $SO(4)$ which stabilizes the first basis vector 1 of $\mathbb{H}$). Hence, by Theorem 2.4 there exists $g \in \mathbb{H}$ of unit norm such that $h^{-1}\, Tv = gvg^{-1}$ for all $v \in \mathbb{H}$. Hence $Tv = hgvg^{-1}$.

Finally we prove that the kernel of the homomorphism consists of 2 elements. Suppose that g_1, g_2 are quaternions of unit norm such that $g_1 v g_2^{-1} = v$ for all $v \in \mathbb{H}$. For $v := 1$ this yields $g_1 = g_2$. Now use the result on the kernel in Theorem 2.4. □

Ex. 2.6 Let $\mathbb{R}^{1,n}$ be the vector space $\mathbb{R}^{n+1}$ with elements $x = (x_0, x_1, \ldots, x_n)$ and indefinite (Lorentzian) inner product $[x, y] := x_0 y_0 - x_1 y_1 - \ldots - x_n y_n$. Let $H_+^{1,n}$ be the set $\{x \in \mathbb{R}^{1,n} \mid [x, x] = 1,\ x_0 > 0\}$ (the upper sheet of a two-sheeted hyperboloid). Let $O(1, n)$ consist of all $T \in GL(n+1, \mathbb{R})$ acting on $\mathbb{R}^{1,n}$ such that $[Tx, Tx] = [x, x]$ for all $x \in \mathbb{R}^{1,n}$. Let $SO_0(1, n)$ consist of all $T \in O(1, n)$ such that $\det T = 1$ and $T_{0,0} := [Te_0, e_0] > 0$. ($SO_0(1, n)$ is the *Lorentz group* for 1 time dimension and n space dimensions.) Show the following.

(a) $O(1, n)$ and $SO_0(1, n)$ are closed subgroups of $GL(n+1, \mathbb{R})$.

(b) $(T,v) \mapsto Tv\colon SO_0(1,n) \times H_+^{1,n} \to H_+^{1,n}$ is a transitive and continuous group action of $SO_0(1,n)$ on $H_+^{1,n}$, and the stabilizer of e_0 in $SO_0(1,n)$ is $SO(n)$, where $SO(n)$ is embedded in $SO_0(1,n)$ as $T \mapsto \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$.

(c) Every $T \in SO_0(1,n)$ can be written as

$$T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t & 0 & \dots & 0 \\ \sinh t & \cosh t & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

for some $A, B \in SO(n)$ and some $t \geq 0$. Conclude that $SO_0(1,n)$ is connected.
Hint This is analogous to the proof of Proposition 1.14.

(d) $G := O(1,n)$ has 4 connected components and $G_0 = SO_0(1,n)$.

2.7 Proposition Every $T \in GL(n,\mathbb{C})$ can be uniquely written as $T = UH$, where $U \in U(n)$ and H is a positive definite hermitian matrix. This is called the *polar decomposition* of T. The resulting map $T \mapsto (U,H)$ is continuous. So we have homeomorphisms

$(U,H) \mapsto UH$: $U(n) \times \{n \times n \text{ pos. def. hermitian matrices}\} \to GL(n,\mathbb{C})$

$(U,H) \mapsto UH$: $SU(n) \times \{n \times n \text{ pos. def. hermitian matrices of det. } 1\} \to SL(n,\mathbb{C})$

$(O,S) \mapsto OS$: $O(n) \times \{n \times n \text{ pos. def. real symmetric matrices}\} \to GL(n,\mathbb{R})$

$(O,S) \mapsto OS$: $SO(n) \times \{n \times n \text{ pos. def. real symmetric matrices of det. } 1\} \to SL(n,\mathbb{R})$

(the three last homeomorphisms by restriction).

Proof (sketch) First, if H is a positive definite hermitian matrix then define $H^{\frac{1}{2}}$ as the unique positive definite hermitian matrix which has H as its square. The existence and uniqueness of $H^{\frac{1}{2}}$ follow because H and $H^{\frac{1}{2}}$ must have the same eigenvectors while the eigenvalues of $H^{\frac{1}{2}}$ must be the positive square roots of the eigenvalues of H. Also note that, if $T \in GL(n,\mathbb{C})$ then $T^* T$ is positive definite hermitian.

If $T \in GL(n,\mathbb{C})$ can be written as $T = UH$ with $U \in U(n)$ and H a positive definite hermitian matrix, then

$$T^* T = (UH)^* UH = H^* U^* UH = HU^{-1} UH = H^2,$$

so

$$H = (T^* T)^{\frac{1}{2}}, \quad U = T (T^* T)^{-\frac{1}{2}}.$$

Conversely, if $T \in GL(n,\mathbb{C})$ then $T = (T (T^* T)^{-\frac{1}{2}}) (T^* T)^{\frac{1}{2}} = UH$ with U, H as above. Moreover, H is then positive definite hermitian, as we already observed, and U is unitary, since

$$\big(T (T^* T)^{-\frac{1}{2}}\big)^* \big(T (T^* T)^{-\frac{1}{2}}\big) = (T^* T)^{-\frac{1}{2}} T^* T (T^* T)^{-\frac{1}{2}} = I.$$

The continuity of the map $H \mapsto H^{\frac{1}{2}}$ from the set of positive definite hermitian matrices to itself can be most easily seen by using the matrix exponential $\exp\colon A \mapsto \sum_{k=0}^{\infty} A^k/k!$ (A a square matrix) and the inverse of this map, denoted by log. Then it can be shown that exp is a homeomorphism of the space of hermitian matrices onto the set of positive definite matrices, and that $H^{\frac{1}{2}} = \exp(\frac{1}{2}\log(H))$ (H positive definite). We will discuss this exponential map in more detail later.

2.8 Corollary The groups $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$ and $SL(n,\mathbb{R})$ are connected.

Proof This uses Proposition 2.7 and the fact that $U(n)$, $SU(n)$ and $SO(n)$ are connected. This last thing we did not yet prove except for $SO(n)$ and for $SU(2)$. For general $U(n)$ and $SU(n)$ it can be proved by using that there is a transitive action of $U(n)$ and $SU(n)$ on the unit sphere in $\mathbb{C}^n$ with stabilizer of e_1 given by $U(n-1)$ and $SU(n-1)$, respectively. $\square$

2.9 Proposition We have the following correspondences:

$$\begin{array}{rcl}
\mathbb{R}^{1,3} & \longleftrightarrow & \{2\times 2 \text{ hermitian matrices}\} \\
(t,x,y,z) & \longleftrightarrow & \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix} \quad (\mathbb{R}\text{-linear and bijective}) \\
t & = & \frac{1}{2}\,\mathrm{tr}\begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix} \\
t^2-x^2-y^2-z^2 & = & \det\begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix} \\
H_+^{1,3} & \longleftrightarrow & \{2\times 2 \text{ pos. def. matrices of determinant } 1\}
\end{array}$$

2.10 Theorem For $g\in SL(2,\mathbb{C})$ and A a 2×2 hermitian matrix put $T_g(A) := gAg^*$. Under the identification of the space of 2×2 hermitian matrices with $\mathbb{R}^{1,3}$ the map T_g is a linear transformation of $\mathbb{R}^{1,3}$. Then the map $g\mapsto T_g$ is a surjective continuous group homomorphism from $SL(2,\mathbb{C})$ onto $SO_0(1,3)$ with kernel consisting of 2 elements $\pm I$. The map $g\mapsto T_g$ restricted to $g\in SU(2)$ is essentially the double covering homomorphism $SU(2)\to SO(3)$ discussed in Theorem 2.4.

Proof If A is a 2×2 hermitian matrix and $g\in SL(2,\mathbb{C})$ then gAg^* is hermitian and $\det(gAg^*)=\det(A)$. Hence $T_g\in O(1,3)$. The map $g\mapsto T_g$ is easily seen to be a continuous homomorphism. Since $SL(2,\mathbb{C})$ is connected and $SO_0(1,3)$ is the connected component of I in $O(1,3)$, we see that $T_g\in SO_0(1,3)$.

Next we will prove the surjectivity. Put $\Phi\colon (t,x,y,z)\mapsto \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix}$ for the map from $\mathbb{R}^{1,3}$ to the 2×2 hermitian matrices. In view of Exercise 2.6(c) the surjectivity will follow if for any A of the form

$$A=\begin{pmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad A=\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \quad (B\in SO(3))$$

we can find an element $g\in SL(2,\mathbb{C})$ such that $T_g(\Phi(v))=\Phi(Av)$ for all $v\in\mathbb{R}^{1,3}$. For A of the first type we can take $g:=\begin{pmatrix} e^{\frac{1}{2}s} & 0 \\ 0 & e^{-\frac{1}{2}s} \end{pmatrix}$. For A of the second type we can take a suitable $h\in SU(2)$ since then $h^*=h^{-1}$ and

$$h\begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix}h^* = t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\,h\begin{pmatrix} ix & z+iy \\ -z+iy & -ix \end{pmatrix}h^{-1}.$$

Now apply Theorem 2.4.

Finally we prove that the kernel consists of 2 elements. Suppose that $g \in SL(2,\mathbb{C})$ such that $gAg^* = A$ for all 2×2 hermitian matrices A. For $A := I$ this yields $gg^* = I$. Hence $g \in U(2) \cap SL(2,\mathbb{C}) = SU(2)$. Now apply again Theorem 2.4. $\square$

Ex. 2.11 Consider the map $g \mapsto T_g$, defined in Theorem 2.10, with g restricted to $SL(2,\mathbb{R})$. Identify $\mathbb{R}^{1,2}$ (as a subspace of $\mathbb{R}^{1,3}$) with the space of real symmetric 2×2 matrices (as a subspace of the space of hermitian 2×2 matrices) as follows:

$$(t,x,y,0) \quad \longleftrightarrow \quad \begin{pmatrix} t+x & y \\ y & t-x \end{pmatrix}.$$

(a) Show that $T_g(A) = A$ for all $g \in SL(2,\mathbb{R})$ if $A := \begin{pmatrix} 0 & -iz \\ +iz & 0 \end{pmatrix}$ and $z \in \mathbb{R}$. Conclude that the map $g \mapsto T_g$ restricts to a continuous homomorphism from $SL(2,\mathbb{R})$ into $SO_0(1,2)$ with kernel consisting of 2 elements $\pm I$.

(b) Show that the homomorphism in (a) is surjective from $SL(2,\mathbb{R})$ onto $SO_0(1,2)$.

Ex. 2.12 Construct from the surjective two-to-one homomorphisms $SU(2) \to SO(3)$ and $SU(2) \times SU(2) \to SO(4)$ a surjective two-to-one homomorphism from $SO(4)$ onto $SO(3) \times SO(3)$ and show that its kernel is $\pm I$.

Ex. 2.13 Let $\Sigma := \mathbb{C} \cup \{\infty\}$ (the extended complex plane). For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$ and $z \in \Sigma$ put

$$g.z := \frac{az+b}{cz+d}.$$

Such a transformation is called a *Möbius transformation* of Σ.

(a) Show that the map $(g,z) \mapsto g.z$ defines a group action of G on Σ.

(b) Show that the set $\{g \in GL(2,\mathbb{C}) \mid g.z = z \text{ for all } z \in \Sigma\}$ equals the subgroup of nonzero multiples of I in $GL(2,\mathbb{C})$.

(c) Conclude from (b) that the map sending g to the transformation $z \mapsto g.z$ is a two-to-one homomorphism from $SL(2,\mathbb{C})$ onto the group of Möbius transformations of Σ and that this homomorphism has kernel $\pm I$.
Remark Put $PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\{\pm I\}$. Then $PSL(2,\mathbb{C})$ is isomorphic to the group of all Möbius transformations of Σ.

(d) Conclude from (b) that the map sending g to the transformation $z \mapsto g.z$ with g restricted to $SU(2)$ is a two-to-one homomorphism from $SU(2)$ into the group of Möbius transformations of Σ and that this homomorphism has kernel $\pm I$.

Ex. 2.14 Below we give a bijection between the unit sphere S^2 in $\mathbb{R}^3$ and the extended complex plane Σ (which thus can be considered as a sphere and is called *Riemann sphere*).

By this bijection we can transplant certain orthogonal transformations of S^2 to certain Möbius transformations of Σ as follows:

$$\begin{array}{rcl} S^2 & \longleftrightarrow & \Sigma \\ (x,y,z) & \longleftrightarrow & \dfrac{x+iy}{1-z} \\ (x\,\cos 2\theta - y\,\sin 2\theta, x\,\sin 2\theta + y\,\cos 2\theta, z) & \longleftrightarrow & \dfrac{e^{i\theta}\,\frac{x+iy}{1-z} + 0}{0\,\frac{x+iy}{1-z} + e^{-i\theta}} \\ (x\,\cos 2\theta - z\,\sin 2\theta, y, x\,\sin 2\theta + z\,\cos 2\theta) & \longleftrightarrow & \dfrac{\cos\theta\,\frac{x+iy}{1-z} + \sin\theta}{-\sin\theta\,\frac{x+iy}{1-z} + \cos\theta} \end{array}$$

Show that the above correspondences extend to an isomorphism from $SO(3)$ onto the group of Möbius transformations of the form $g \mapsto g.z$ with $g \in SU(2)$.

Ex. 2.15 The group $SO_0(1,3)$ acts on $\mathbb{R}^{1,3}$. By restriction of this action $SO_0(1,3)$ acts on the *forward light cone* $\{(t,x,y,z) \in \mathbb{R}^{1,3} \mid x^2+y^2+z^2 = t^2,\ t > 0\}$, and also on the *space of light rays* consisting of all half-lines $\{\lambda(t,x,y,z) \mid \lambda > 0\}$ with (t,x,y,z) in the forward light cone. There is a bijection between the space of light rays and the unit sphere S^2 in $\mathbb{R}^3$ given by:

$$\{\lambda(t,x,y,z) \mid \lambda > 0\} \quad \longleftrightarrow \quad t^{-1}(x,y,z).$$

Thus the action of $SO_0(1,3)$ on the space of light rays can be transplanted to an action of this group on S^2 (which may be called the *celestial sphere* in this context). This defines a homomorphism $T \mapsto \widetilde{T}$ from $SO_0(1,3)$ into the group of bijective transformations (in fact homeomorphisms) of S^2.

(a) Show that $\widetilde{T}\,(x,y,z) = \tau^{-1}(\xi,\eta,\zeta)$, where $T(1,x,y,z) = (\tau,\xi,\eta,\zeta)$.

(b) Show that $\widetilde{T} = T$ if $T \in SO(3)$, with $SO(3)$ considered as subgroup of $SO_0(1,3)$.

(c) Show that $\widetilde{T}\,(x,y,z) = (\cosh t + z\,\sinh t)^{-1}\,(x, y, \sinh t + z\,\cosh t)$ if

$$T = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}.$$

(d) Transplant by the bijection $S^2 \leftrightarrow \Sigma$ (see Exercise 2.14) the transformation $\widetilde{T}$ of S^2 to the transformation $\widetilde{\widetilde{T}}$ of Σ. Show that $T \mapsto \widetilde{\widetilde{T}}$ maps $SO(3)$ isomorphically onto $SU(2)/\{\pm I\}$ and that $\widetilde{\widetilde{T}}(w) = \begin{pmatrix} e^{\frac{1}{2}t} & 0 \\ 0 & e^{-\frac{1}{2}t} \end{pmatrix}.w$ $(w \in \Sigma)$ if T is as in (c). Conclude that the map $T \mapsto \widetilde{\widetilde{T}}$ is an isomorphism from $SO_0(1,3)$ onto $PSL(2,\mathbb{C}\,)$.

Ex. 2.16 Let $\mathbb{C}^{1,1}$ be the vector space $\mathbb{C}^2$ with elements $z = (z_1,z_2)$ and indefinite (Lorentzian) hermitian inner product $[z,w] := z_1\,\overline{w_1} - z_2\,\overline{w_2}$. Let $SU(1,1)$ consist of all $T \in SL(2,\mathbb{C}\,)$ acting on $\mathbb{C}^{1,1}$ such that $[Tz,Tz] = [z,z]$ for all $z \in C^{1,1}$. Show the following.

(a) $SU(1,1)$ is a closed subgroup of $SL(2,\mathbb{C})$ and of $GL(2,\mathbb{C})$.

(b) $SU(1,1)$ consists of all matrices $\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$ such that $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$.

(c) Let $g_0 := \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, then $g_0^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. Show that there is an isomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \quad : \quad SL(2,\mathbb{R}) \longleftrightarrow SU(1,1)$$

such that

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = g_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_0^{-1}.$$

Show that under this correspondence

$$a = \mathrm{Re}\,(\alpha + \overline{\beta}), \quad b = \mathrm{Im}\,(\alpha + \overline{\beta}),$$
$$c = -\mathrm{Im}\,(\alpha - \overline{\beta}), \quad d = \mathrm{Re}\,(\alpha - \overline{\beta}),$$

and

$$\alpha = \tfrac{1}{2}\big(a + d + i(b - c)\big), \quad \beta = \tfrac{1}{2}\big(a - d - i(b + c)\big).$$

(d) Let T be the unit circle and D be the open unit disk, both as subsets of $\mathbb{C}$. Show that there is a homeomorphism $\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \leftrightarrow (u, w) \colon SU(1,1) \leftrightarrow T \times D$ given by $u = \alpha/|\alpha|$, $w = \beta/\alpha$ with inversion formula $\alpha = u/\sqrt{1 - |w|^2}$, $\beta = uw/\sqrt{1 - |w|^2}$.

2.17 Remark By a *one-parameter group* in a topological group G we mean a continuous homomorphism $\alpha\colon \mathbb{R} \to G$. We are mainly interested in the two cases that α is injective or that α has kernel $2\pi\mathbb{Z}$. In the latter case we may view the one-parameter group as a continuous injective homomorphism from the circle group T into G. In these two cases we then have an isomorphic image of the group $\mathbb{R}$ or $\mathbb{T}$ as subgroup of G. Below we list three different one-parameter subgroups $\theta \mapsto u_\theta\colon T \to SL(2,\mathbb{R})$, $t \mapsto a_t\colon \mathbb{R} \to SL(2,\mathbb{R})$ and $x \mapsto n_x\colon \mathbb{R} \to SL(2,\mathbb{R})$, the corresponding one-parameter subgroups in $SU(1,1)$ (same notation but with tilde), and the traces of the elements u_θ, a_t, n_x of the various one-parameter subgroups of $SL(2,\mathbb{R})$. (The corresponding element of $SU(1,1)$ then has the same trace.)

$$u_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \widetilde{u}_\theta := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \qquad \mathrm{tr} = 2\cos\theta \in [-2, 2];$$
$$a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad \widetilde{a}_t := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \qquad \mathrm{tr} = 2\cosh t \in [2, \infty);$$
$$n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad \widetilde{n}_x := \begin{pmatrix} 1 + \frac{1}{2}ix & -\frac{1}{2}ix \\ \frac{1}{2}ix & 1 - \frac{1}{2}ix \end{pmatrix}, \qquad \mathrm{tr} = 2.$$

2.18 We call two elements g_1, g_2 in a group G *conjugate* to each other (within G) if $hg_1h^{-1} = g_2$ for some $h \in G$. The set of all elements in G conjugate to some fixed element of G is called a *conjugacy class*.

Proposition Each $g \in SL(2,\mathbb{R})$ is conjugate to some element u_θ or $\pm a_t$ or $\pm n_x$ (with notation as in Remark 2.17).

Proof Let $g \in SL(2,\mathbb{R})$. We can factorize the characteristic polynomial of g as $\det(zI - g) = (z-\lambda)(z-\mu)$. Here $\lambda, \mu \in \mathbb{C}$. Since the matrix g is real, either λ and μ are real or they are non-real but complex conjugate to each other. Furthermore, $1 = \det g = \lambda\mu$, hence $\lambda \neq 0$ and $\mu = \lambda^{-1}$. Furthermore, $\operatorname{tr} g = \lambda + \mu = \lambda + \lambda^{-1}$. After possibly replacing g by $-g$, we may assume that either $\lambda > 0$ or $\lambda = e^{i\theta}$ with $0 \neq \theta \in (-\pi, \pi)$. If $\lambda \neq 1$ then $\lambda \neq \lambda^{-1}$ and there are eigenvectors $v, w \in \mathbb{C}^2$, mutually independent, such that $g\,v = \lambda\,v$ and $g\,w = \lambda^{-1}\,w$. If $\lambda = 1$ then g has at least one eigenvector v with eigenvalue 1. We distinguish several cases:

(a) $\lambda = e^t$ with $0 \neq t \in \mathbb{R}$. Then $\operatorname{Re} v$ and $\operatorname{Im} v$ have both eigenvalue e^t (since g is a real matrix and e^t is real), hence these two vectors must be proportional, and similarly for $\operatorname{Re} w$ and $\operatorname{Im} w$. So without loss of generality we may assume that the eigenvectors v and w are in $\mathbb{R}^2$. After possibly multiplying w with a suitable nonzero real constant, we may also assume that $\det(v, w) = 1$. Then $g = h a_t h^{-1}$, with $h \in SL(2,\mathbb{R})$ sending the standard basis vectors e_1, e_2 to v, w, respectively.

(b) $\lambda = e^{i\theta}$ with $0 \neq \theta \in (-\pi, \pi)$. Then $g\,v = e^{i\theta}\,v$ and, taking complex conjugates, $g\,\overline{v} = e^{-i\theta}\,\overline{v}$. Hence v and $\overline{v}$ are mutually independent in $\mathbb{C}^2$, so $\operatorname{Re} v$ and $\operatorname{Im} v$ are mutually independent in $\mathbb{R}^2$ and

$$g \operatorname{Re} v = \cos\theta \operatorname{Re} v - \sin\theta \operatorname{Im} v, \quad g \operatorname{Im} v = \sin\theta \operatorname{Re} v + \cos\theta \operatorname{Im} v.$$

After possibly multiplying $\operatorname{Re} v$ and $\operatorname{Im} v$ with the same positive constant and after possibly changing θ into $-\theta$, we have $\det(\operatorname{Re} v, \operatorname{Im} v) = 1$. Then $g = h u_{-\theta} h^{-1}$, with $h \in SL(2,\mathbb{R})$ sending e_1, e_2 to $\operatorname{Re} v, \operatorname{Im} v$, respectively.

(c) $\lambda = 1$. If $g = I$ then we are done. Otherwise, there exists $v \in \mathbb{C}^2 \backslash \{0\}$ such that $g\,v = v$ and v is unique up to a nonzero complex constant factor. By a same reasoning as in (a) we may assume that $v \in \mathbb{R}^2$. Choose $w \in \mathbb{R}^2$ such that $\det(v, w) = 1$. Then $g = h n_x h^{-1}$ for some $x \in \mathbb{R}$, with $h \in SL(2,\mathbb{R})$ sending e_1, e_2 to v, w, respectively.

Ex. 2.19 The trace of $g \in SL(2,\mathbb{R})$ is invariant on conjugacy classes. Hence, by Remark 2.17, no conjugacy is possible between u_θ and a_t, or between u_θ and n_x, or between a_t and n_x unless both elements are equal to I. Now prove the following:

(a) Let $0 \neq \theta \in (-\pi, \pi)$. Then u_θ is not conjugate to $u_{-\theta}$. In fact, if some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ is conjugate to u_θ then b has the same sign as $\sin\theta$.

(b) Let $0 \neq t \in \mathbb{R}$. Then a_t is conjugate to a_{-t}.

(c) Let $x, y \in \mathbb{R}\backslash\{0\}$. Then x is conjugate to y iff x and y have the same sign. In fact, if some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ is conjugate to n_x then b has the same sign as x.

2.20 Theorem (summary of the above results) Let $g \in SL(2,\mathbb{R})$. Then g is conjugate within $SL(2,\mathbb{R})$ to one and only one of the following elements:

(i) ($|\operatorname{tr} g| > 2$, hyperbolic elements) $\pm a_t$ $(t > 0)$;

(ii) ($|\operatorname{tr} g| < 2$, elliptic elements) $\pm u_\theta$ $(0 < \theta < \pi)$;

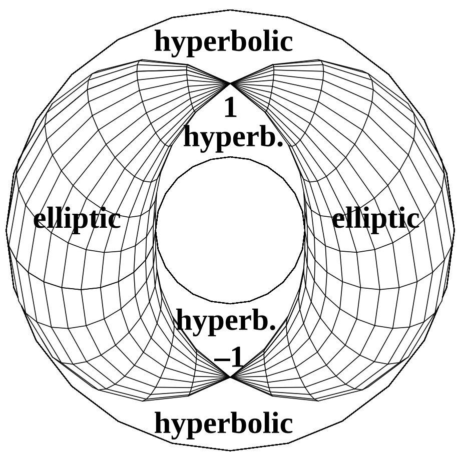
(iii) ($|\operatorname{tr} g| = 2$), trivial elements $\pm I$ and parabolic elements $\pm n_1$, $\pm n_{-1}$.

2.21 *Visualisation of conjugacy classes of $SL(2,\mathbb{R})$.*
We now draw the conjuagacy classes of $SL(2,\mathbb{R})$, or equivalently of $SU(1,1)$, where we identify $SU(1,1)$ topologically with $T\times D$ (see Exercise 2.16(d)), and visualize $T\times D$ as the ring in the (x,y,z)-plane obtained by rotating the disk $\{(x,y) \mid (x-2)^2+y^2<1\}$ around the y-axis in the (x,y,z)-space. Thus, x,y,z are expressed in terms of $(u,w)\in T\times D$ by

$$x+iz=u(2+\operatorname{Re} w),\quad y=\operatorname{Im} w.$$

Conjugation of $\begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix}\in SU(1,1)$ with respect to $\widetilde{u}_\theta$ shows that $\begin{pmatrix}\alpha & e^{2i\theta}\beta\\ e^{-2i\theta}\overline{\beta} & \overline{\alpha}\end{pmatrix}$ is in the same conjugacy class. In terms of the parameters $(u,w)\in T\times D$ this means that $(u,e^{2i\theta}w)$ is in the same conjugacy class for all θ. Hence, in the picture of our ring in $\mathbb{R}^3$ the conjugacy classes are rotation symmetric around the central circle $\{(x,z)\mid x^2+z^2=4\}$ in the ring, so the conjugacy classes can already be visualized in the (x,z)-plane between the two concentric circles of radii 1 and 3. I refer to a 3-d suggestion of the conjugacy classes in the picture below (resembling the picture at p.57 of G. Segal's lecture notes). Here the x-direction points vertically on the page and the y-direction is perpendicular to the page. I will also make available a Maple worksheet `sl2r.mws`, which gives both a 2-d version in the (x,z)-plane and a 3d-version. Note the sausage-like figure in the picture below. The boundary of the sausage has four connected components, corresponding to the four parabolic conjugacy classes. The interior of the sausage has two connected components (marked B in the picture), which correspond to elliptic equivalence classes containing $\widetilde{u}_\theta$ for $0<\theta<\pi$ respectively $-\pi<\theta<0$. The exterior of the sausage also has two connected components which correspond to hyperbolic conjugacy classes contaning a_t respectively $-a_t$.

It is also interesting to consider how the one-parameter subgroups are situated within the ring. The elliptic one-parameter subgroup $\theta\mapsto\widetilde{u}_\theta$ is the central circle of the ring. Its conjugates do not remain in the (x,z)-plane, but still pass through I and $-I$. The hyperbolic one-parameter subgroup $t\mapsto\widetilde{a}_t$ passes vertically through I. Its conjugates are in region A and pass through I, hitting the boundary of the ring as $t\to\pm\infty$, but not remaining in the (x,z)-plane. The parabolic one-parameter groups are within the two top components of the boundary of the sausage, pass through I and hit the boundary of the ring as $x\to\pm\infty$ in points where the boundary of the sausage meets the boundary of the ring, but they do not remain in the (x,z)-plane.



a geometric vieuw of $SL(2,\mathbb{R})$

Chapter 3. Differentiable manifolds

3.1 Definition Let X be a topological space.

(a) A *chart of dimension* d for X is a pair (U,ψ) with U an open subset of X en $\psi\colon U\to\mathbb{R}^d$ such that $\psi(U)$ is open in $\mathbb{R}^d$ and $\psi\colon U\to\psi(U)$ is a homeomorphism.

(b) Twee charts (U_1,ψ_1) and (U_2,ψ_2) for X are called C^∞*-compatible* if they have the same dimension d and if the map $\psi_2\circ\psi_1^{-1}\colon\psi_1(U_1\cap U_2)\to\psi_2(U_1\cap U_2)$ is a C^∞-diffeomorphism. (Observe that $\psi_1(U_1\cap U_2)$ is open in $\psi_1(U_1)$, hence it is also open in $\mathbb{R}^d$.)

(c) A C^∞*-atlas of dimension* d for X is a collection $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ of mutually C^∞-compatible charts of dimension d for X (A an index set) such that $\bigcup_{\alpha\in A}U_\alpha=X$.

(d) A C^∞-manifold of dimension d is a topological space X equipped with a C^∞-atlas of dimension d. Moreover we will always assume that X as a topological space is Hausdorff and that *the second axiom of countability* is satisfied (i.e., there is a countable collection $\{W_i\}_{i\in I}$ of open subsets of X such that each open subset U of X is a union of a subcollection of the W_i 's).

(e) Let $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ be a C^∞-atlas for X. A chart (U,ψ) for X is called C^∞*-compatible* with the atlas if (U,ψ) is C^∞-compatible with each chart (U_α,ψ_α) in the atlas. Two C^∞-atlases for X are called *equivalent* if each chart in the first atlas is C^∞-compatible with each chart in the second atlas. Two C^∞-manifolds built on the same topological space X and having equivalent atlases will not be distinguished from each other as C^∞-manifolds.

Ex. 3.2 Let X be a C^∞-manifold with atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$.

a) Let $\alpha\in A$, let U be an open subset of U_α and let $\psi:=\psi_\alpha|_U$. Show that (U,ψ) is a chart which is C^∞-compatible with the atlas.

b) Let (U,ψ) and (V,χ) be two charts for X which are C^∞-compatible with the atlas. Show that the two charts are also C^∞-compatible with each other.

Ex. 3.3 Show the following.

(a) Let X be a C^∞-manifold of dimension d with atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$. Let $W\subset X$. Then W is open iff for all $\alpha\in A$ the set $\psi_\alpha(W\cap U_\alpha)$ is open in $\mathbb{R}^d$ (or, equivalently, open in $\psi_\alpha(U_\alpha)$).

(b) Let X be a set. Let be given a collection $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$, where A is an index set, $\{U_\alpha\}_{\alpha\in A}$ a collection of subsets of X which cover X, $\psi_\alpha\colon U_\alpha\to V_\alpha$ a bijective map from U_α onto some open subset V_α of $\mathbb{R}^d$, and where the following holds: For each $\alpha,\beta\in A$ the sets $\psi_\alpha(U_\alpha\cap U_\beta)$ and $\psi_\beta(U_\alpha\cap U_\beta)$ are open in $\mathbb{R}^d$ and $\psi_\beta\circ\psi_\alpha^{-1}\colon\psi_\alpha(U_\alpha\cap U_\beta)\to\psi_\beta(U_\alpha\cap U_\beta)$ is a C^∞ diffeomorphism. Then we can define a topology on X by calling $W\subset X$ open if for all $\alpha\in A$ the set $\psi_\alpha(W\cap U_\alpha)$ is open in $\mathbb{R}^d$. If this topology is moreover Hausdorff and satisfies the second axiom of countability then X with atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ is a C^∞-manifold.

3.4 Example (taken from Segal, p.69)
One can cover the sphere

$$S^2 := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$

by six open sets $U_1, \ldots, U_6$, where

U_1 consists of the points where $x > 0$,
U_2 consists of the points where $x < 0$,
U_3 consists of the points where $y > 0$,

and so on. There are obvious charts (U_i, ψ_i) with $\psi_i\colon U_i \to V_i \subset \mathbb{R}^2$; for example $\psi_1(x,y,z) = (y,z)$. These charts are C^∞-compatible — e.g. the transition map ψ_{13} is given by

$$\psi_{13}(y,z) = ((1-y^2-z^2)^{\frac{1}{2}}, z)$$

They make S^2 into a C^∞-manifold. Another chart belonging to the same atlas is the one given by stereographic projection from the north pole $N := (0,0,1)$ to the equatorial plane $z=0$. This is the homeomorphism from $U := S^2\backslash\{N\}$ to $\mathbb{R}^2$ defined by

$$\psi(x,y,z) := (x/(1-z), y/(1-z).$$

Ex. 3.5 Work out the details of Example 3.4. Show also that the chart using sterographic projection form the north pole and a similar one using sterographic projection from the south pole together already determine S^2 as a C^∞-manifold.

3.6 Definition A k-dimensional (regularly embedded) C^∞-submanifold of $\mathbb{R}^n$ is a subset $X \subset \mathbb{R}^n$ such that for each $x \in X$ there is an open neighbourhood U of x in $\mathbb{R}^n$ and a C^∞-diffeomorphism $\psi\colon U \to V$ onto an open cube V in $\mathbb{R}^n$ parallel to the coordinate axes and including 0 such that $\psi(x) = 0$ and $\psi(U \cap X) = \{y \in V \mid y_{k+1} = \ldots = y_n = 0\}$.
Often X as above is just called a *submanifold* of $\mathbb{R}^n$.

With ψ, U and V as above the map $x \mapsto (\psi_1(x), \ldots, \psi_k(x))\colon U \cap X \to \{(y_1, \ldots, y_k) \in \mathbb{R}^k \mid (y_1, \ldots, y_k, 0, \ldots, 0) \in V\}$ is a chart for X and all charts of this type for X are C^∞-compatible and make X into a k-dimensional C^∞-manifold.

3.7 Proposition A set $X \subset \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$ iff for all $x \in X$ there is an open neighbourhood U of x in $\mathbb{R}^n$ and a C^∞-map $f\colon U \to \mathbb{R}^{n-k}$ such that $f(x) = 0$, the $(n-k) \times n$ matrix $f'(x)$ has (maximal) rank $n-k$ and $X \cap U = \{y \in U \mid f(y) = 0\}$.

3.8 Example $O(n)$ is a submanifold of the space $M_n(\mathbb{R})$ of real $n \times n$ matrices. This can be shown either by Proposition 3.7 or by Definition 3.6.

First Proof $O(n) = \{A \in M_n(\mathbb{R}) \mid A^t A - I = 0\}$. Put $f(A) := A^t A - I$, then f is a C^∞-map from $M_n(\mathbb{R})$ into $\mathrm{Sym}_n(\mathbb{R})$, the space of real $n \times n$ symmetric matrices, which is a $\frac{1}{2}n(n+1)$ dimensional real vector space. So we have to show that for each $A \in O(n)$ the linear map $f'(A)$ from $M_n(\mathbb{R})$ to $\mathrm{Sym}_n(\mathbb{R})$ has (maximal) rank $\frac{1}{2}n(n+1)$. We find that $f'(A)B = A^t B + B^t A$ $(A, B \in M_n(\mathbb{R}))$. Let $A \in O(n)$. Then $f'(A)\,B = 0$ iff $B^t = -A^{-1}BA^{-1}$. This is also equivalent to $(JB)^t = -(JA)^{-1}(JB)(JA)^{-1}$, where J is the $n \times n$ diagonal matrix with $-1, 1, \ldots, 1$ on the main diagonal. Note that $J \in O(n)$ and $\det J = -1$. By a simple geometric argument, for each $A \in SO(n)$ there exists $A^{\frac{1}{2}} \in SO(n)$ such that $(A^{\frac{1}{2}})^2 = A$. Also put $A^{-\frac{1}{2}} := (A^{\frac{1}{2}})^{-1}$. Thus, if $A \in SO(n)$ then $f'(A)\,B = 0$ iff $(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t = -A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. If $JA \in SO(n)$ then $f'(A)\,B = 0$ iff $((JA)^{-\frac{1}{2}}(JB)(JA)^{-\frac{1}{2}})^t = -(JA)^{-\frac{1}{2}}(JB)(JA)^{-\frac{1}{2}}$. In both cases we see that the null space of $f'(A)$ has the same dimension as the space of real $n \times n$ anti-symmetric matrices, i.e., dimension $\frac{1}{2}n(n-1)$. Hence $f'(A)$ has rank $n^2 - \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$. □

Second Proof Let $A, B \in M_n(\mathbb{R})$. Note that the following three statements are equivalent:

(a) $\det(A + I) \neq 0$ and $B = (A - I)(A + I)^{-1}$.

(b) $(I - B)(I + A) = 2I$.

(c) $\det(I - B) \neq 0$ and $A = (I - B)^{-1}(I + B)$.

Also, if these three equivalent conditions hold then check that $A^t = A^{-1}$ iff $B^t = -B$. Put $U := \{A \in M_n(\mathbb{R}) \mid \det(A + I) \neq 0\}$, $V := \{B \in M_n(\mathbb{R}) \mid \det(I - B) \neq 0\}$. These are open subsets of $M_n(\mathbb{R})$. Let $\psi(A) := (A - I)(A + I)^{-1}$. Then $\psi\colon U \to V$ is a C^∞ diffeomorphism. Also ψ maps $U \cap O(n)$ onto V intersected with the space of real $n \times n$ antisymmetric matrices. Thus for each $T \in O(n)$ with $\det(T + I) \neq 0$ we can find open neighbourhood and map as required in Definition 3.6. If $T \in O(n)$ and $\det(T + I) = 0$ then we take as neigbourhood of T the set $\{A \in M_n(\mathbb{R}) \mid T^{-1}A \in U\}$ and we take the map $A \mapsto (T^{-1}A - I)(T^{-1}A + I)^{-1}$. □

3.9 Example (taken from Segal, p.70)
A good example of a manifold which does not arise naturally as a subset of Euclidean space is the projective space $\mathbb{P}(\mathbb{R}^n)$, which consists of all lines through the origin in $\mathbb{R}^n$. A point of $\mathbb{P}(\mathbb{R}^n)$ is represented by n homogeneous coordinates $(x_1, \ldots, x_n)$, not all zero, and $(x_1, \ldots, x_n)$ represents the same point as $(\lambda x_1, \ldots, \lambda x_n)$ if $\lambda \neq 0$. If U_n is the part of $\mathbb{P}(\mathbb{R}^n)$ consisting of points with $x_n \neq 0$ then we have a bijection $\psi_n\colon U_n \to \mathbb{R}^{n-1}$ given by

$$\psi_n(x_1, \ldots, x_n) = (x_1 x_n^{-1}, \ldots, x_{n-1} x_n^{-1}).$$

Obviously, $\mathbb{P}(\mathbb{R}^n)$ is covered by n such sets $U_1, \ldots, U_n$ with bijections $\psi_i\colon U_i \to \mathbb{R}^{n-1}$. One readily checks that they define a C^∞-atlas. Notice that in situations like this we do not need to define a topology on $\mathbb{P}(\mathbb{R}^n)$ explicitly: the atlas provides it with a topology which makes it a manifold (see Exercise 3.3(b)).

Ex. 3.10 Work out the details of Example 3.9

3.11 Example (taken from Segal, p.71)
Only slightly more general is the case of the *Grassmannian* $\mathrm{Gr}_k(\mathbb{R}^n)$, which is the set of all k-dimensional vector subspaces of $\mathbb{R}^n$. A point W of $\mathrm{Gr}_k(\mathbb{R}^n)$ is represented by a $n \times k$ matrix x of rank k, whose columns form a basis for W. In this case x and $x\lambda$ represent

the same point if λ is an invertible $k \times k$ matrix. Thus we have an equivalence relation $x \sim y$ if $x = y\lambda$ for some $\lambda \in GL(k, \mathbb{R})$. We write $\widetilde{x}$ for the equivalence class of x. Let S be any subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, where $i_1 < i_2 < \ldots < i_k$. Let $\{j_1, \ldots, j_{n-k}\}$ be the complementary subset, with $j_1 < j_2 < \ldots < j_{n-k}$. Let $x_S := (x_{i_p,q})_{p,q=1,\ldots,k}$. Let $U_S := \{\widetilde{x} \in Gr_k(\mathbb{R}^n) \mid x_S \in GL(k, \mathbb{R})\}$ and $\psi_S \colon \widetilde{x} \mapsto \big((x x_S^{-1})_{j_p,q}\big)_{p=1,\ldots,n-k;\ q=1,\ldots,k} : U_S \to M_{n-k,k}(\mathbb{R})$. Here $M_{n-k,k}(\mathbb{R})$ is the space of real $(n-k) \times k$ matrices. Note that U_S and ψ_S are well-defined (independently of the choice of the representative x of $\widetilde{x}$), and that the sets U_S cover $\mathrm{Gr}_k(\mathbb{R}^n)$. Let α_S be the $n \times n$ permutation matrix which sends the standard basis vector e_p to e_{i_p} if $1 \le p \le k$ and to $e_{j_{p-k}}$ if $k+1 \le p \le n$. Then, for all $y \in M_{n-k,k}(\mathbb{R})$ we have

$$\psi_S^{-1}(y) = \widetilde{x}, \quad \text{where} \quad x := \alpha_S \begin{pmatrix} 1 \\ y \end{pmatrix}.$$

Here $\begin{pmatrix} 1 \\ y \end{pmatrix}$ is a $(k + (n-k)) \times k$ block matrix. Now let $\xi \in U_S \cap U_T$ and put $y := \psi_S(\xi)$, $z := \psi_T(\xi)$, $x := \alpha_S \begin{pmatrix} 1 \\ y \end{pmatrix}$. Then $\xi = \widetilde{x}$ and $x = \alpha_T \begin{pmatrix} 1 \\ z \end{pmatrix} \lambda$ for some $\lambda \in GL(k, \mathbb{R})$. Hence

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \lambda = \alpha_T^{-1} \alpha_S \begin{pmatrix} 1 \\ y \end{pmatrix}.$$

Write $\alpha_T^{-1}\alpha_S$ as a $(k + (n-k)) \times (k + (n-k))$ block matrix $\alpha_T^{-1}\alpha_S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\lambda = a + by$, $z\lambda = c + dy$. Hence $z = (c+dy)(a+by)^{-1}$, by which we have found the explicit expression for the transition map

$$\psi_T^{-1} \circ \psi_S\colon \quad y \mapsto (c+dy)(a+by)^{-1}\colon \quad \psi_S(U_S \cap U_T) \to \psi_T(U_S \cap U_T),$$

and see that it is C^∞.

3.12 Definition Let X and Y be C^∞-manifolds with atlases $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ resp. $\{(V_\beta, \chi_\beta)\}_{\beta \in B}$.

(a) A map $\phi\colon X \to Y$ is called a *C^∞-map* if ϕ is continuous and if for each $\alpha \in A$ and $\beta \in B$ the map $\chi_\beta \circ \phi \circ \psi_\alpha^{-1}\colon \psi_\alpha(U_\alpha \cap \phi^{-1}(V_\beta)) \to \chi_\beta(V_\beta)$ is a C^∞-map.

(b) We call $\phi\colon X \to Y$ a *C^∞-diffeomorphism* (or briefly *diffeomorphism*) if ϕ is bijective and ϕ and ϕ^{-1} are C^∞-maps.

(c) In the special case of a C^∞-map $\phi\colon X \to \mathbb{R}$ we call ϕ a *C^∞-function* on the manifold X.

(d) Another special case of a C^∞-map is given by a *C^∞-curve* on a C^∞-manifold X. Such a curve can be defined as a C^∞-map $\phi\colon I \to X$, where $I \subset \mathbb{R}$ is an open interval.

Ex. 3.13 Let X be a C^∞-manifold. Let $\phi\colon X \to \mathbb{R}$. Show that ϕ is C^∞ iff for each $\alpha \in A$ the map $\phi \circ \psi_\alpha^{-1}\colon \psi_\alpha(U_\alpha) \to \mathbb{R}$ is a C^∞-map (so it is not necessary to require separately that ϕ is continuous).

3.14 Remark
Let X and Y be C^∞-manifolds with atlases $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ resp. $\{(V_\beta,\chi_\beta)\}_{\beta\in B}$. let $\phi\colon X\to Y$. Then ϕ is C^∞ iff the following holds:
For each $x\in X$ and for some $\beta\in B$ with $\phi(x)\in V_\beta$ there is $\alpha\in A$ and an open neighbourhood $U\subset U_\alpha$ of x such that
(i) $\phi(U)\subset V_\beta$;
(ii) the map $\chi_\beta\circ\phi\circ\psi_\alpha^{-1}\colon\psi_\alpha(U)\to\chi_\beta(V_\beta)$ is C^∞.

3.15 Remark Let X_1, X_2 and X_3 be C^∞-manifolds. Let $\phi_1\colon X_1\to X_2$ and $\phi_2\colon X_2\to X_3$ be C^∞-maps. Then $\phi_2\circ\phi_1\colon X_1\to X_3$ is a C^∞-map.

3.16 Definition Let X be a C^∞-manifold with atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ and let Y be a C^∞-manifold with atlas $\{(V_\beta,\chi_\beta)\}_{\beta\in B}$. Then the *product manifold* of X and Y is defined as the set $X\times Y$ equipped with the product topology and with the atlas $\{(U_\alpha\times V_\beta,\psi_\alpha\times\chi_\beta)\}_{\alpha\in A,\,\beta\in B}$.

3.17 Definition A *Lie group* is a set G which has the structure of a group and also the structure of a C^∞-manifold such that the map $(x,y)\mapsto xy\colon G\times G\to G$ is a C^∞-map.

3.18 Example $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ are examples of Lie groups. Note that they are open subsets of $M_n(\mathbb{R})$ resp. $M_n(\mathbb{C})$, so one chart (embedding identically into $\mathbb{R}^{n^2}$ resp. $\mathbb{R}^{2n^2}$) suffices for defining their structure as C^∞-manifold.

3.19 Proposition In a Lie group G the map $x\mapsto x^{-1}\colon G\to G$ is C^∞.
Proof We will first prove that $x\mapsto x^{-1}$ is C^∞ when restricted to a sufficiently small neighbourhood of e. Let G have dimension d. Let (U,ψ) be a chart for G such that $e\in U$, $\psi(e)=(0,\ldots,0)$. For $x\in U$ write $\psi(x)=(x_1,\ldots,x_d)$. Put

$$F(x_1,\ldots,x_d,y_1,\ldots,y_d):=\psi(\psi^{-1}(x_1,\ldots,x_d)\psi^{-1}(y_1,\ldots,y_d)),$$

which is well-defined for $(x_1,\ldots,x_d)$ and $(y_1,\ldots,y_d)$ sufficiently close to $(0,\ldots,0)$ and which yields a C^∞ map there. Then

$$F(x_1,\ldots,x_d,0,\ldots,0)=(x_1,\ldots,x_d)\quad\text{and}\quad\frac{\partial F(x_1,\ldots,x_d,0,\ldots,0)}{\partial(x_1,\ldots,x_d)}=I.$$

Hence the implicit function theorem yields for $(x_1,\ldots,x_d)$ and $(y_1,\ldots,y_d)$ sufficiently close to $(0,\ldots,0)$ that the equation $F(x_1,\ldots,x_d,y_1,\ldots,y_d)=0$ has for each $(x_1,\ldots,x_d)$ a unique solution $(y_1,\ldots,y_d)=(y_1(x_1,\ldots,x_d),\ldots,y_d(x_1,\ldots,x_d))$, and that this is a C^∞-function. Then it follows that $x^{-1}=\psi^{-1}(y_1(x_1,\ldots,x_d),\ldots,y_d(x_1,\ldots,x_d))$ for x close to e.

In order to show that $x\mapsto x^{-1}\colon G\to G$ is C^∞ everywhere on G fix $g\in G$ and let x be in a sufficiently small neighbourhood of g. Then the map $x\mapsto x^{-1}$ can be factorized as the composition of three maps which are all C^∞:

$$x\quad\mapsto\quad g^{-1}x\quad\mapsto\quad(g^{-1}x)^{-1}\quad\mapsto\quad(g^{-1}x)^{-1}g^{-1}=x^{-1}.$$ □

3.20 Proposition If G is a subgroup of $GL(n,\mathbb{R})$ and also a submanifold of $M_n(\mathbb{R})$ then G is a Lie group.

Proof Suppose that G has dimension k. Let $g_1, g_2 \in G$ and $g_3 := g_1 g_2$, so $g_3 \in G$. Because G is a submanifold, there are for $j = 1, 2, 3$ diffeomorphisms $\psi_j : U_j \to V_j$ with U_j open in $GL(n, \mathbb{R})$, $g_j \in U_j$, V_j open cube in $\mathbb{R}^{n^2}$ parallel to the coordinate axes, $\psi_j(g_j) = (0, \ldots, 0)$, $\psi_j(U_j \cap G) = \{x \in V_j \mid x_{k+1} = \ldots = x_{n^2} = 0\}$. Because multiplication in G is a C^∞-map, the map $(x_1, \ldots, x_{n^2}, y_1, \ldots, y_{n^2}) \mapsto \psi_3(\psi_1^{-1}(x_1, \ldots, x_{n^2})\psi_2^{-1}(y_1, \ldots, y_{n^2}))$, well-defined for x and y sufficiently close to 0 in $\mathbb{R}^{n^2}$, is C^∞. By restriction we will then get a map $(x_1, \ldots, x_k, 0, \ldots, 0, y_1, \ldots, y_k, 0, \ldots, 0) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)$, where $z_1, \ldots, z_k$ depends in a C^∞ way on $(x_1, \ldots, x_k, y_1, \ldots, y_k)$. Since we are dealing here with charts for the submanifold G (see Definition 3.6) in terms of which we have written the product on G, we conclude that G is a Lie group. □

3.21 Example By Proposition 3.20 and Example 3.8 we see that $O(n)$ is a Lie group.

3.22 Proposition If G is a subgroup of $GL(n, \mathbb{C})$ and also a submanifold of the $(2n^2)$-dimensional real vector space $M_n(\mathbb{C})$ of complex $n \times n$ matrices, then G is a Lie group.

Proof The proof is similar to the above proof of Proposition 3.20. We leave it as an exercise.

3.23 Definition Let G, H be Lie groups and $f : G \to H$. Then f is called a *homomorphism* (of Lie groups) if f is a both a C^∞-map and a group homomorphism. Furthermore, f is called an *isomorphism* (of Lie groups) if f is a group isomorphism such that f and f^{-1} are C^∞-maps.

3.24 Definition A *linear Lie group* is a Lie group which for certain n is isomorphic to a certain subgroup and submanifold G of $GL(n, \mathbb{C})$.

3.25 Definition Let X be a C^∞-manifold of dimension d with atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ and let $x \in X$. Then a *tangent vector* v to X at x is a choice of a vector $v_\alpha \in \mathbb{R}^d$ for each $\alpha \in A$ for which $x \in U_\alpha$, where the vectors v_α must be related to each other by $v_\beta = (\psi_\beta \circ \psi_\alpha^{-1})'(\psi_\alpha(x))\, v_\alpha$. (Recall that the derivative of a C^∞-map from an open subset U of $\mathbb{R}^d$ to $\mathbb{R}^d$ taken at a certain point of U is a linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$.) Thus a tangent vector v is already determined by a choice of v_α for one $\alpha \in A$, since this determines v_β for other β. If v, w are tangent vectors at x and $\lambda, \mu \in \mathbb{R}$ then $\lambda v + \mu w$ is defined as the tangent vector such that for all α we have $(\lambda v + \mu w)_\alpha = \lambda v_\alpha + \mu w_\alpha$. In this way, the *tangent space* $T_x X$ of all tangent vectors at x becomes a real vector space of dimension d, and for each α we have a linear bijection $v \to v_\alpha : T_x X \to \mathbb{R}^n$.

Alternatively we can describe tangent vectors at $x \in X$ as equivalence classes of C^∞ maps $\gamma : (-\varepsilon, \varepsilon) \to X$ with $\gamma(0) = x$ (this gives a C^∞-curve in X through x). We call γ_1 and γ_2 *equivalent* if for some chart (U_α, ψ_α) with $x \in U_\alpha$ (and hence for all such charts) we have $(\psi_\alpha \circ \gamma_1)'(0) = (\psi_\alpha \circ \gamma_2)'(0)$. There is a one-to-one correspondence between such equivalence classes with representative γ and tangent vectors v at x. The correspondence is given by $v_\alpha = (\psi_\alpha \circ \gamma_1)'(0)$. If γ and v are related in this way then we write $v = \gamma'(0)$.

If X is a k-dimensional submanifold of $\mathbb{R}^n$, if $x \in X$ and if U, V, ψ are as in Definition 3.6 then the k-dimensional vector space $T_x X$ can be seen as a linear subspace of $\mathbb{R}^n$, namely as the set of vectors $(\psi^{-1})'(0)\,(x_1,\ldots,x_k,0,\ldots,0)$, where $(x_1,\ldots,x_k)\in\mathbb{R}^k$. Also, if $\gamma\colon(-\varepsilon,\varepsilon)\to X$ is a C^∞-map with $\gamma(0)=x$ then we can obtain the tangent vector $\gamma'(0)$ by differentiating γ at 0 as a C^∞-map from $(-\varepsilon,\varepsilon)$ to $\mathbb{R}^n$, so it is not needed here to obtain $\gamma'(0)$ via a chart.

3.26 We present now a third approach to tangent vectors, namely as linear functionals on the space of C^∞-functions on a C^∞-manifold X. We will need the following four results, formualated as exercises.

Ex. 3.27 Let $0<a<b$. Construct $f\in C^\infty(\mathbb{R}^d)$ such that $f(x)\ge 0$ for all $x\in\mathbb{R}^d$, $f(x)=1$ if $|x|\le a$ and $f(x)=0$ if $|x|\ge b$.
Hint Start with a C^∞-function g on $\mathbb{R}$ given by

$$g(x):=\exp\left(\frac{1}{x-b}-\frac{1}{x-a}\right)\quad\text{if } a<x<b\qquad\text{and}\quad g(x):=0\quad\text{otherwise.}$$

Ex. 3.28 Let X be a C^∞-manifold, let $x\in X$ and let V be an open neighbourhood of x. Let $f\in C^\infty(V)$. Show that there exists a C^∞-function g on X such that g coincides with f on a certain open neighbourhood $U\subset V$ of x.
Hint Work with a chart around x and use Exercise 3.27.

Ex. 3.29 Let f be a C^∞-function defined on an open ball B_r with radius $r>0$ about 0 in $\mathbb{R}^d$. Show that

$$f(x)=f(0)+\sum_{j=1}^d x_j\,g_j(x)\quad(x\in B_r)$$

for some $g_j\in C^\infty(B_r)$ with $g_j(0)=(D_jf)(0)$.
Hint Use that $G(1)-G(0)=\int_0^1 G'(t)\,dt$ with $G(t):=f(tx)$.

Ex. 3.30 Let X be a C^∞-manifold of dimension d and let $x\in X$. Let $f\in C^\infty(X)$. Let (U,ψ) be a chart for X such that $x\in U$ and $\psi(x)=0$. Then there is an open neighbourhood $V\subset U$ of x and there are functions $g_1,\ldots,g_d,h_1,\ldots,h_d\in C^\infty(X)$ such that $h_1(x)=\ldots=h_d(x)=0$ and $g_j(x)=(D_j(f\circ\psi^{-1}))(0)$ $(j=1,\ldots d)$ and

$$f=f(x)+\sum_{j=1}^d h_j\,g_j\quad\text{on } V. \tag{3.1}$$

Hint Apply Exercise 3.29 to the function $f\circ\psi^{-1}$ restricted to some ball about 0 within $\psi(U)$. Then use Exercise 3.28.

3.31 Definition (third definition of tangent vector and tangent space)
Let X be a C^∞-manifold and let $x\in X$. A *tangent vector to X at x* is a linear map $v\colon C^\infty(X)\to\mathbb{R}$ such that

$$v(fg)=f(x)\,v(g)+g(x)\,v(f),\quad f,g\in C^\infty(X).$$

As in Definition 3.25 we call the real linear space of all tangent vectors the *tangent space to x at X* and we denote it by T_xX.

3.32 Proposition The tangent vectors to X at x, as defined by Definition 3.31, form a real vector space with the same dimension d as X. If (U,ψ) is a chart for X such that $x\in U$ and $\psi(x)=0$ then T_xX consists of all v of the form

$$v(f)=\sum_{k=1}^d a_k\,\frac{\partial}{\partial y_k}f(\psi^{-1}(y))\Big|_{y=0} \tag{3.2}$$

and the map $(a_1,\ldots,a_d)\mapsto v\colon\mathbb{R}^d\to T_xX$ is a linear bijection.

Proof It is easily seen that each v of the form (3.2) is a tangent vector to X at x according to Definition 3.31, and that the map $(a_1,\ldots,a_d)\mapsto v\colon\mathbb{R}^d\to T_xX$ is a linear injection. We have to show that the map is surjective.
First observe that $v(1)=0$ if $v\in T_xX$ and 1 is the function identically equal to 1 on X.
Next observe that $v(f)=0$ is $v\in T_xX$ and $f\in C^\infty(X)$ is identically equal to 0 on some neighbourhood of x. Hence $v(f)$ only depends on the restriction of f to an arbitrary small neighbourhood of x.
Next apply Exercise 3.30. Let (U,ψ) be a chart for X such that $x\in U$ and $\psi(x)=0$. So, for each $f\in C^\infty(X)$ there are functions g_j and h_j with properties as in Exercise 3.30, so we can write (3.1) as

$$v(f)=f(x)\,v(1)+\sum_{j=1}^d h_j(x)v(g_j)+\sum_{j=1}^d g_j(x)v(h_j)=\sum_{j=1}^d v(h_j)(D_j(f\circ\psi^{-1}))(0).$$

So (3.2) is satisfied with $a_k:=v(h_k)$. □

Ex. 3.33 Let X be a C^∞-manifold of dimension d, let $x\in X$ and let (U,ψ) and $V,\chi)$ be two charts for X such that $x\in U\cap V$ and $\psi(x)=0=\chi(x)$. As a consequence of Proposition 3.32 we have linear bijections $v\leftrightarrow(a_1,\ldots,a_d)\leftrightarrow(b_1,\ldots,b_d)\colon T_xX\leftrightarrow\mathbb{R}^d\leftrightarrow\mathbb{R}^d$ such that, for all $f\in C^\infty(X)$,

$$v(f)=\sum_{i=1}^d a_i\,\frac{\partial}{\partial y_i}f(\psi^{-1}(y))\Big|_{y=0}=\sum_{j=1}^d b_j\,\frac{\partial}{\partial z_j}f(\chi^{-1}(z))\Big|_{z=0}$$

Show that

$$a_i=\sum_{j=1}^d b_j\,\frac{\partial(\psi\circ\chi^{-1})_i(z)}{\partial z_j}\Bigg|_{z=0}.$$

Conclude that Definition 3.31 for tangent vector is equivalent to Definition 3.25.

Chapter 4. The exponential mapping

4.1 Definition Let $\mathbb{F} := \mathbb{R}$ or $\mathbb{C}$). An *algebra* over $\mathbb{F}$ is a linear space $\mathcal{A}$ over $\mathbb{F}$ together with a bilinear map $(a,b) \mapsto ab: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. The algebra is called *associative* if $(ab)c = a(bc)$ for all $a,b,c \in \mathcal{A}$ and it is called *commutative* if $ab = ba$ for all $a,b \in \mathcal{A}$.

4.2 Definition A *Lie algebra* over $\mathbb{F}$ is a linear space $\mathfrak{g}$ over $\mathbb{F}$ together with a bilinear map $(X,Y) \mapsto [X,Y]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for all $X,Y,Z \in \mathfrak{g}$ the following holds:
(a) $[X,Y] = -[Y,X]$ (*anti-commutativity*);
(b) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (*Jacobi identity*).

We call $[X,Y]$ the *Lie bracket* of X and Y. It follows from (a) that $[X,X] = 0$ for all $X \in \mathfrak{g}$.

4.3 Example Every associative algebra $\mathcal{A}$ becomes a Lie algebra with Lie bracket $[X,Y] := XY - YX$. In particular, $M_n(\mathbb{F})$ is an associative algebra over $\mathbb{F}$ with respect to matrix multiplication. Hence it becomes a Lie algebra. We will write $gl(n,\mathbb{F})$ for $M_n(\mathbb{F})$ considered as a Lie algebra.

4.4 Definition A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a linear subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[X,Y] \in \mathfrak{h}$ for all $X,Y \in \mathfrak{h}$. Thus $\mathfrak{h}$, with Lie bracket inherited from $\mathfrak{g}$, becomes a Lie algebra itself.

4.5 Definition A *Lie algebra homomorphism* $\Phi: \mathfrak{g} \to \mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ to a Lie algebra $\mathfrak{h}$ is a linear map Φ such that $[\Phi(X), \Phi(Y)] = \Phi([X,Y])$ for all $X,Y \in \mathfrak{g}$. We call Φ a *Lie algebra isomorphism* if Φ is an invertible Lie algebra homomorphism. Then Φ^{-1} is also a Lie algebra homomorphism.

4.6 (definition and properties of the exponential mapping)

(a) For $A \in M_n(\mathbb{C})$ define $e^A = \exp(A) := \sum_{k=0}^\infty A^k/k!$. Since $\|A^k\| \le \|A\|^k \le R^k$ if $\|A\| \le R$ and since $\sum_{k=0}^\infty R^k/k! < \infty$, the series defining e^A is absolutely convergent, uniformly for A in any compact subset of $M_n(\mathbb{C})$. (Note that convergence in norm and convergence in each matrix entry are equivalent, since we are dealing with a finite dimensional vector space.)

(b) If $AB = BA$ then $e^A e^B = e^{A+B}$. This follows because

$$\left(\sum_{k=0}^\infty \frac{A^k}{k!}\right)\left(\sum_{l=0}^\infty \frac{B^l}{l!}\right) = \sum_{m=0}^\infty \left(\sum_{k,l;\ k+l=m} \frac{A^k B^l}{k!\, l!}\right).$$

Here reordering of terms is allowed because the doubly infinite series on the left-hand side converges absolutely.

(c) In particular, $e^A e^{-A} = I$. Hence $A \mapsto e^A$ maps $M_n(\mathbb{C})$ into $GL(n,\mathbb{C})$ and, by restriction, $M_n(\mathbb{R})$ into $GL(n,\mathbb{R})$.

(d) The map $A \mapsto e^A: M_n(\mathbb{C}) \to GL(n,\mathbb{C})$ is complex analytic, hence real analytic, hence C^∞. This follows by the uniform convergence on compacta of $\sum_{k=0}^\infty A^k/k!$. By restriction, the map $A \mapsto e^A: M_n(\mathbb{R}) \to GL(n,\mathbb{R})$ is real analytic, hence C^∞.

(e) If $A \in M_n(\mathbb{C})$ then $e^{(z+w)A} = e^{zA}e^{wA}$ $(z,w \in \mathbb{C})$. The map $z \mapsto e^{zA}: \mathbb{C} \to GL(n,\mathbb{C})$ is complex analytic. The map $t \mapsto e^{tA}: \mathbb{R} \to GL(n,\mathbb{C})$ is real analytic. Furthermore, $\frac{d}{dt}\, e^{tA}\big|_{t=0} = A$.

(f) The derivative at 0 of the map $A \mapsto e^A: M_n(\mathbb{C}) \to GL(n,\mathbb{C})$ equals $\mathrm{id}: M_n(\mathbb{C}) \to M_n(\mathbb{C})$. The derivative at 0 of the map $A \mapsto e^A: M_n(\mathbb{R}) \to GL(n,\mathbb{R})$ equals $\mathrm{id}: M_n(\mathbb{R}) \to M_n(\mathbb{R})$.

(g) There is an open neighbourhood U of 0 in $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$) and an open neighbourhood V of I in $GL(n,\mathbb{C})$ (resp. $GL(n,\mathbb{R})$) such that $\exp: U \to V$ is a C^∞ diffeomorphism. (Use (f) and the inverse function theorem.)

(h) If $B \in GL(n,\mathbb{C})$ and $A \in M_n(\mathbb{C})$ then $B \exp(A)\, B^{-1} = \exp(BAB^{-1})$.

(i) If $A \in M_n(\mathbb{C})$ has eigenvalues $\lambda_1,\ldots,\lambda_n$ (i.e., $\det(\lambda I - A) = (\lambda-\lambda_1)\ldots(\lambda-\lambda_n)$) then e^A has eigenvalues $e^{\lambda_1},\ldots,e^{\lambda_n}$.

(j) If $A \in M_n(\mathbb{C})$ then $\det(e^A) = e^{\mathrm{tr}\, A}$.

(k) For $B \in M_n(\mathbb{C})$ with $\|B-I\| < 1$ (which implies that B is invertible) define $\log(B) := -\sum_{k=1}^\infty k^{-1}(I-B)^k$. This series is absolutely convergent, uniformly for $\|B-I\| < R$ if $R < 1$. Hence the map $\log: \{B \in GL(n,\mathbb{C}) \mid \|B-I\| < 1\} \to M_n(\mathbb{C})$ is complex analytic, hence real analytic, hence C^∞.

(l) $\exp(\log(B)) = B$ if $\|B-I\| < 1$. This follows by the following reasoning. Since $e^{\log z} = z$ for $z \in \mathbb{C}$ outside $(-\infty, 0]$ and since this can written as an identity of convergent power series in $1-z$ for $|1-z| < 1$, we have an identity of formal power series in $I-B$ given by

$$\sum_{k=0}^\infty \frac{1}{k!}\left(-\sum_{l=1}^\infty \frac{(I-B)^l}{l}\right)^k = B.$$

Now the left-hand side of this last identity wiil converge absolutely for $\|B-I\| < 1$, since

$$\sum_{k=0}^\infty \frac{1}{k!}\left(\sum_{l=1}^\infty \frac{\|B-I\|^l}{l}\right)^k = \exp(-\log(1-\|B-I\|)) = \frac{1}{1-\|B-I\|} < \infty.$$

(m) We can choose the open neighbourhoods U and V discussed in (g) such that $V \subset \{B \in GL(n,\mathbb{C}) \mid \|B-I\| < 1\}$ and the inverse of the C^∞ diffeomorphism $\exp: U \to V$ is given by $\log: V \to U$.

Ex. 4.7 Work out the details of §4.6.

4.8 Proposition Let $A, B \in M_n(\mathbb{C})$ and $t \in \mathbb{R}$. Then

$$\log(e^{tA}e^{tB}) = t(A+B) + \tfrac{1}{2}t^2[A,B] + \mathcal{O}(|t|^3) \quad \text{as } |t| \to 0.$$

Proof For $|t| \to 0$ we have

$$e^{tA}e^{tB} - I = t(A+B) + \tfrac{1}{2}t^2(A^2 + 2AB + B^2) + \mathcal{O}(|t|^3) = \mathcal{O}(|t|).$$

Hence

$$(e^{tA}e^{tB} - I)^2 = t^2(A^2 + AB + BA + B^2) + \mathcal{O}(|t|^3).$$

Hence

$$\log(e^{tA}e^{tB}) = (e^{tA}e^{tB} - I) - \tfrac{1}{2}(e^{tA}e^{tB} - I)^2 + \mathcal{O}(|t|^3) = t(A+B) + \tfrac{1}{2}t^2[A,B] + \mathcal{O}(|t|^3). \quad \square$$

Ex. 4.9 Let $A, B \in M_n(\mathbb{C})$ and $t \in \mathbb{R}$. Show that the following holds as $|t| \to 0$.

(a) $e^{tA}e^{tB}e^{-tA} - I = tB + \frac{1}{2}t^2(B^2 + 2[A,B]) + \mathcal{O}(|t|^3)$;

(b) $e^{tA}e^{tB}e^{-tA}e^{-tB} - I = t^2[A,B] + \mathcal{O}(|t|^3)$.

4.10 Example (taken from Segal, p.72)
If $G = O(n)$ as submanifold of $M_n(\mathbb{R})$ then T_IG is the $\frac{1}{2}n(n-1)$-dimensional vector space S of all skew-symmetric matrices and $T_gG = gS = Sg$ for all $g \in G$.

Proof For any skew matix A the matrix e^{tA} is orthogonal, so $\gamma(t) = ge^{tA}$ defines a path $\gamma\colon \mathbb{R} \to G$ such that $\gamma(0) = g$ and $\gamma'(0) = gA$. Conversely, if $\gamma\colon (-\varepsilon, \varepsilon) \to G$ is a path such that $\gamma(0) = g$ then by differentiating $\gamma(s)\,(\gamma(s))^t$ we find

$$\gamma'(0)\,g^t + g\,(\gamma'(0))^t = 0,$$

which shows that $g^{-1}\gamma'(0)$ is skew-symmetric, i.e. that $T_gG \subset gS$. $\square$

Ex. 4.11 Show the following. If $G = U(n)$ then $T_IG = \{A \in M_n(\mathbb{C}) \mid A^* = -A\}$.

4.12 Definition Let X, Y be C^∞-manifolds and $f\colon X \to Y$ a C^∞-map. Let $x \in X$. Then the *differential* of f at x is the linear map $df_x\colon T_xX \to T_{f(x)}Y$ defined by the rule: $df_x(v) = (f \circ \gamma)'(0)$ if $\gamma\colon (-\varepsilon, \varepsilon) \to X$ is C^∞ with $\gamma(0) = x$ and $v = \gamma'(0)$.

4.13 Proposition Let $G \subset GL(n, \mathbb{C})$ be a linear Lie group. Put $\mathfrak{g} := T_IG$ (thus $\mathfrak{g}$ is a real linear subspace of $M_n(\mathbb{C})$. Then:

(a) Let $g \in G$ and $A \in M_n(\mathbb{C})$. Then $A \in \mathfrak{g}$ iff $gA \in T_gG$ iff $Ag \in T_gG$.

(b) $\exp(\mathfrak{g}) \subset G$

(c) If $A, B \in \mathfrak{g}$ then $[A, B] := AB - BA \in \mathfrak{g}$. So $\mathfrak{g}$ is a real Lie subalgebra of $gl(n, \mathbb{C})$. It is called the *Lie algebra of* G.

Proof (a) Let $g \in G$ and $A \in \mathfrak{g}$. Then there is a C^∞ map $t \mapsto \alpha(t)\colon (-\varepsilon, \varepsilon) \to GL(n, \mathbb{C})$ such that $\alpha(t) \in G$ for all t, $\alpha(0) = I$ and $\alpha'(0) = A$. Now put $\beta(t) := g\,\alpha(t)$ and $\gamma(t) := \alpha(t)\,g$. Then β, γ are C^∞ maps such that $\beta(t), \gamma(t) \in G$ for all t, $\beta(0) = g = \gamma(0)$, and $\beta'(0) = gA$, $\gamma'(0) = Ag$. This shows the implications in one direction. The implications in the other direction follow because $\dim T_gG = \dim \mathfrak{g}$.

(b) Let $A \in \mathfrak{g}$, $\alpha(t) := \exp(tA)$. Then $\alpha(t) \in GL(n,\mathbb{C})$ for all t and it is a solution of the system of o.d.e.'s

$$\alpha'(t) = A\,\alpha(t), \quad \alpha(0) = I.$$

Since G is a submanifold of $M_n(\mathbb{C})$ (say of real dimension m), there is an open neighbourhood U of I in $GL(n,\mathbb{C})$, an open cube V in $\mathbb{R}^{2n^2}$ parallel to the coordinate axes, and a C^∞ diffeomorphism $\psi: U \to V$ such that $\psi(I) = 0$ and $\psi(U \cap G) = \mathbb{R}^m \cap V$, where $\mathbb{R}^m$ is the linear subspace of all $y \in \mathbb{R}^{2n^2}$ for which $y_{m+1} = \ldots = y_{2n^2} = 0$. Put $\beta(t) := \psi(\alpha(t))$. Then, for $|t|$ small enough, $\beta(t)$ lies in V and is a solution of the system of o.d.e.'s

$$\beta'(t) = f(\beta(t)), \quad \beta(0) = 0,$$

where

$$f(y) := d\psi_{\psi^{-1}(y)}(A\,\psi^{-1}(y)) \quad (y \in V).$$

Note that $f: V \to \mathbb{R}^{2n^2}$ is C^∞. Also, if $y \in \mathbb{R}^m \cap V$ then $\psi^{-1}(y) \in U \cap G$, so $A\,\psi^{-1}(y) \in T_{\psi^{-1}(y)}G$, so finally $f(y) \in T_y\mathbb{R}^m = \mathbb{R}^m$. So we can solve in $\mathbb{R}^m$ the system of o.d.e.'s

$$\gamma_i'(t) = f_i(\gamma(t)), \quad \gamma_i(0) = 0 \qquad (i = 1, \ldots, m),$$

and then $\gamma(t) := (\gamma_1(t), \ldots, \gamma_m(t), 0, \ldots, 0)$ will solve the same system of o.d.e.'s as $\beta(t)$. By uniqueness of the solution (since f is C^∞), we have $\beta(t) = \gamma(t)$, hence $\psi(\exp(tA)) = \gamma(t) \in \mathbb{R}^m \cap V$ for $|t|$ small. Hence $\exp(tA) \in U \cap G$ for $|t|$ small. Since G is a group we have for $t \in \mathbb{R}$ and $k \in \mathbb{N}$ that $\exp(tA) = (\exp(k^{-1}tA))^k \in G$ if $k^{-1}|t|$ is sufficiently small.

(c) Let $A, B \in \mathfrak{g}$ and $t \in \mathbb{R}$. Then $e^{tA} \in G$ by (b) and $e^{tA}Be^{-tA} \in T_IG$ by twice applying (a). Hence $e^{tA}(AB - BA)e^{-tA} = \frac{d}{dt}(e^{tA}Be^{-tA}) \in T_IG$. So, for $t = 0$ we get that $AB - BA \in T_IG$. □

4.14 Proposition If $G \subset GL(n,\mathbb{C})$ is a linear Lie group, then G is a closed subgroup of $GL(n,\mathbb{C})$.

Proof Let $g_0 \in GL(n,\mathbb{C})$, $g_0 = \lim_{k\to\infty} g_k$ with $g_k \in G$. We have to prove that $g_0 \in G$. Let G have dimension m. As before, there exist an open neighbourhood U of I in $GL(n,\mathbb{C})$, an open cube V in $\mathbb{R}^{2n^2}$ parallel to the coordinate axes, and a C^∞ diffeomorphism $\psi: U \to V$ such that $\psi(I) = 0$ and $\psi(U \cap G) = \mathbb{R}^m \cap V$. Then $\mathbb{R}^m \cap V$ is a closed subset of V, hence $U \cap G$ is a closed subset of U. Let U_0 be an open neighbourhood of I in U such that $\overline{U_0} \subset U$, where $\overline{U_0}$ means closure of U_0 in $M_n(\mathbb{C})$. By the convergence of the sequence (g_k) there will be some $N \in \mathbb{N}$ such that $g_l^{-1}g_k \in U_0$ if $k, l \geq N$. Thus, if $k \geq N$ then $g_N^{-1}g_k \in U_0 \cap G \subset U \cap G$. Since $g_N^{-1}g_k \in U_0$, its limit in $M_n(\mathbb{C})$ will be in the closure of U_0, so $g_N^{-1}g_0 \in \overline{U_0} \subset U$. So $g_N^{-1}g_0 \in U$ is the limit in U of the sequence $(g_N^{-1}g_k)$ lying in the closed subset $G \cap U$ of G. Hence $g_N^{-1}g_0$ is itself in this closed subset $G \cap U$. Hence $g_N^{-1}g_0 \in G$, so $g_0 \in g_NG$, so $g_0 \in G$. □

4.15 Proposition Suppose that G is a subgroup of $GL(n,\mathbb{C})$, that $\mathfrak{g}$ is a real linear subspace of $M_n(\mathbb{C})$ and that there are open neighbourhoods U of 0 in $M_n(\mathbb{C})$ and V of I in $GL(n,\mathbb{C})$ such that $\exp: U \to V$ is a C^∞ diffeomorphism and $G \cap V = \exp(\mathfrak{g} \cap U)$. Then $G \subset GL(n,\mathbb{C})$ is a linear Lie group and $T_IG = \mathfrak{g}$.

Proof Let $\dim \mathfrak{g} = m$. By possibly shrinking the neighbourhoods U, V and by taking suitable real cordinates on $M_n(\mathbb{C})$ we can arrange that U is an open cube in $\mathbb{R}^{2n^2}$ parallel to the coordinate axes and that $\exp^{-1}(G \cap V) = U \cap \mathbb{R}^m$. Now consider arbitrary $g \in G$. Let $\ell_{g^{-1}}$ denote left multiplication by g^{-1} on $GL(n, \mathbb{C})$. Then gV is an open neigbourhood of g in $GL(n, \mathbb{C})$, U is a cubic neighbourhood of 0 in $M_n(\mathbb{C})$ and $\exp \circ \ell_{g^{-1}} : gV \to U$ is a C^∞ diffeomorphism sending g to 0 and satisfying $(\exp^{-1} \circ \ell_{g^{-1}})(G \cap gV) = U \cap \mathbb{R}^m$. So G is a submanifold of $M_n(\mathbb{C})$ and a subgroup of $GL(n, \mathbb{C})$, so G is a linear Lie group in $GL(n, \mathbb{C})$. Finally, by the remark at the end of Definition 3.25 and by §4.6(f) we get that $T_I G = d\exp_0(\mathbb{R}^m) = \mathrm{id}(\mathbb{R}^m) = \mathbb{R}^m = \mathfrak{g}$. □

4.16 We can use Proposition 4.15 in order to prove that the matrix groups introduced in §1.2 of these notes are linear Lie groups and to find their Lie algebras. Instead of $O(n)$ and $SO(n)$ we will also write $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$, respectively.

Proposition Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The following table lists examples of linear Lie groups G in $GL(n, \mathbb{C})$ with corresponding Lie algebras $\mathfrak{g}$.

G	$\mathfrak{g}$
$GL(n, \mathbb{F})$	$gl(n, \mathbb{F}) := M_n(\mathbb{F})$
$SL(n, \mathbb{F}) := \{T \in GL(n, \mathbb{F}) \mid \det T = 1\}$	$sl(n, \mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid \mathrm{tr}\, A = 0\}$
$O(n, \mathbb{F}) := \{T \in GL(n, \mathbb{F}) \mid T T^t = I\}$	$o(n, \mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid A^t = -A\}$
$SO(n, \mathbb{F}) := \{T \in O(n, \mathbb{F}) \mid \det T = 1\}$	$o(n, \mathbb{F})$
$U(n) := \{T \in GL(n, \mathbb{C}) \mid TT^* = I\}$	$u(n) := \{A \in M_n(\mathbb{C}) \mid A^* = -A\}$
$SU(n) := \{T \in U(n) \mid \det T = 1\}$	$su(n) = \{A \in u(n) \mid \mathrm{tr}\, A = 0\}$

Proof We give the proof for $G := SU(n)$ and leave the other cases as an exercise. Let U be an open neigbourhood of 0 in $M_n(\mathbb{C})$ and V an open neigbourhood of I in $GL(n, \mathbb{C})$ such that $\exp : U \to V$ is a diffeomorphism. After possibly shrinking U, we may assume that $|\mathrm{tr}\, A| < 2\pi$ if $A \in U$. Let $U_0 := U \cap (-U) \cap U^* \cap (-U^*)$. Then U_0 is an open neighbourhood of 0 in $M_n(\mathbb{C})$ such that $U_0 = -U_0 = U_0^* = -U_0^*$ and $|\mathrm{tr}\, A| < 2\pi$ if $A \in U_0$. Put $V_0 := \exp(U_0)$. We know that $SU(n)$ is a subgroup of $GL(n, \mathbb{C})$ and that $su(n)$ is a real linear subspace of $M_n(\mathbb{C})$. If $A \in su(n) \cap U_0$ then $\exp A \in SU(n) \cap V_0$ (this is straightforward). Conversely, if $T \in SU(n) \cap V_0$ then $T = \exp A$ for a unique $A \in U_0$. Then $(\exp A)^* = (\exp A)^{-1} = \exp(-A)$. Hence $A^* = -A$ because A^* and $-A$ are both in U_0. Also, $1 = \det(\exp A) = e^{\mathrm{tr}\, A}$, hence $\mathrm{tr}\, A = 2\pi i n$ for some $n \in \mathbb{Z}$, hence $\mathrm{tr}\, A = 0$ because $|\mathrm{tr}\, A| < 2\pi$. So $\exp(su(n) \cap U_0) = SU(n) \cap V_0$ and we can apply Proposition 4.15. □

Ex. 4.17 Give the proof for the other cases of Proposition 4.16.

Ex. 4.18 Let J_n be the $(2n) \times (2n)$ matrix given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Prove that the *complex symplectic group* $Sp(n, \mathbb{C}) := \{T \in GL(2n, \mathbb{C}) \mid J_n T^t J_n^{-1} = T^{-1}\}$ and the *unitary symplectic group* $Sp(n) := Sp(n, \mathbb{C}) \cap U(2n)$ are linear Lie groups with Lie algebras $sp(n, \mathbb{C}) := \{A \in M_{2n}(\mathbb{C}) \mid J_n A J_n^{-1} = -A^t\}$ resp. $sp(n) := sp(n, \mathbb{C}) \cap u(2n)$.

Chapter 5. The Lie algebra of an abstract Lie group

5.1 Definition Let X be a C^∞-manifold. Suppose that for each $x \in X$ a tangent vector $V_x \in T_x X$ is given. Then we call V a *vector field* on X. By Definition 3.31 a vector field V on X defines for each C^∞-function f on X a new function Vf on X by the rule

$$(Vf)(x) := V_x(f) \qquad (x \in X).$$

The vector field V is called a *C^∞-vector field* if $Vf \in C^\infty(X)$ for all $f \in C^\infty(X)$.

5.2 Remark By Definition 3.31 a vector field V on X can be equivalently defined as a linear map V from $C^\infty(X)$ into the linear space of all real-valued functions on X such that

$$V(fg) = V(f)\,g + f\,V(g) \quad \text{for all } f, g \in C^\infty(X).$$

The vector field V is moreover C^∞ iff moreover V maps $C^\infty(X)$ into itself.

5.3 Remark By Propositions 3.13 and 3.32 a C^∞-vector field V on X can be equivalently described as a map V from $C^\infty(X)$ into the linear space of all real-valued functions on X such that for each chart (U, ψ) on X we can write V in the form

$$(Vf)(\psi^{-1}(x)) = \sum_{j=1}^{d} a_j(x)\,\frac{\partial}{\partial x_j} f(\psi^{-1}(x)) \qquad (x \in \psi(U),\ f \in C^\infty(X))$$

for certain functions $a_j \in C^\infty(\psi(U))$.

5.4 Proposition Let V and W be C^∞-vector fields on a C^∞-manifold X. Then the map $[V, W]\colon f \mapsto V(Wf) - W(Vf)\colon C^\infty(X) \to C^\infty(X)$ is again a C^∞-vector field. In particular, the space of C^∞-vector fields forms a Lie algebra with respect to this Lie bracket.

Proof Let (U, ψ) be a chart for X and write

$$(Vf)(\psi^{-1}(x)) = \sum_{j=1}^{d} a_j(x)\,\frac{\partial}{\partial x_j} f(\psi^{-1}(x)),$$
$$(Wf)(\psi^{-1}(x)) = \sum_{j=1}^{d} b_j(x)\,\frac{\partial}{\partial x_j} f(\psi^{-1}(x))$$

for $f \in C^\infty(X)$. Then we find that

$$([V, W]\,f)(\psi^{-1}(x)) = \sum_{j=1}^{d} c_j(x)\,\frac{\partial}{\partial x_j} f(\psi^{-1}(x)),$$

with

$$c_j(x) = \sum_{i=1}^{n} \Big(a_i(x)\,\frac{\partial}{\partial x_i} b_j(x) - b_i(x)\,\frac{\partial}{\partial x_i} a_j(x)\Big).$$

It follows from Example 4.3 that the space of C^∞-vector fields forms a Lie algebra. □

5.5 Remark In Definition 4.12 we defined $d\phi_x: T_xX \to T_{\phi(x)}Y$ where $\phi: X \to Y$ is a C^∞-map between C^∞-manifolds and $x \in X$. The definition of $d\phi_x)(v)$ $(v \in T_xX)$ was given in terms of a C^∞-curve representing the tangent vector v. We can reformulate this by using the interpretation of tangent vectors als linear maps $v: C^\infty(X) \to \mathbb{R}$. Then we obtain:

$$(d\phi_x(v))(f) = v(f \circ \phi) \qquad (v \in T_xX,\ f \in C^\infty(Y)).$$

5.6 Definition For a Lie group G and for $y \in G$ consider the *left multiplication* map $\ell_y: x \mapsto yx: G \to G$. Then clearly ℓ_y is a C^∞-diffeomorphism.

Let G be a Lie group and let $V: C^\infty(X) \to C^\infty(X)$ be a linear map. Then V is called *left invariant* if

$$V(f \circ \ell_x) = (Vf) \circ \ell_x \quad \text{for all } f \in C^\infty(G) \text{ and all } x \in G.$$

In particular, we can thus define left invariance for C^∞-vector fields. It is straightforward to see that the commutator $[V, W]$ of two left invariant C^∞-vector fields V, W on G is again left invariant. Thus the left invariant vector fields on G form a Lie algebra, which we will denote by $\mathrm{Lie}(G)$.

5.7 Proposition There is a linear bijection $V \mapsto V_e: \mathrm{Lie}(G) \to T_eG$ from the space of left invariant C^∞-vector fields on G to the tangent space to G at e. The inverse map associates to $v \in T_eG$ the vector field V given by $V_x = (dl_x)_e\, v$.

In particular, $\mathrm{Lie}(G)$ has finite dimension equal to the dimension of T_eG and of G.

Proof The map $V \mapsto V_e: \mathrm{Lie}(G) \to T_eG$ is linear. If $V \in \mathrm{Lie}(G)$ and $x \in G$ then we have for all $f \in C^\infty(X)$ that

$$V_x(f) = (Vf)(x) = ((Vf) \circ \ell_x)(e) = (V(f \circ \ell_x))(e) = V_e(f \circ \ell_x) = ((d\ell_x)_e\, V_e)(f).$$

Hence $V_x = (d\ell_x)_e\, V_e$.

Conversely, let $v \in T_eG$ and let V be the vector field defined by $V_x = d(l_x)_e\, v$ $(x \in G)$. We have to show that:

(a) V is a C^∞-vector field.
(b) $V_e = v$
(c) V is a left invariant vector field.

For the proof of (a), choose a chart (U, ψ) with $e \in U$ and $\psi(e) = 0$ and note that

$$v(f) = \sum_{j=1}^{d} a_j \frac{\partial}{\partial y_j} f(\psi^{-1}(y))\Big|_{y=0} \quad (f \in C^\infty(G))$$

for certain $a_1, \ldots, a_d \in \mathbb{R}$. Thus, if $f \in C^\infty(G)$ then

$$((dl_x)_e\, v)(f) = v(f \circ \ell_x) = \sum_{j=1}^{d} a_j \frac{\partial}{\partial y_j} f(x\psi^{-1}(y))\Big|_{y=0}.$$

This last expression is a C^∞-function in $x \in G$ since $(x, y) \mapsto f(x\psi^{-1}(y)): G \times \psi(U) \to \mathbb{R}$ is C^∞. Thus $x \mapsto ((dl_x)_e\, v)(f)$ is C^∞ whenever f is C^∞, so the vector field V is C^∞.

The proof of (b) is immediate. For the proof of (c) let $f \in C^\infty(G)$ and $x, y \in G$. Then

$$\begin{aligned}((Vf) \circ \ell_x)(y) &= V_{xy}(f) = ((d\ell_{xy})_e\, v)(f) = ((d\ell_x)_y \circ (d\ell_y)_e)\, v)(f) \\ &= ((d\ell_y)_e)\, v)(f \circ \ell_x) = V_y(f \circ \ell_x) = (V(f \circ \ell_x))(y).\end{aligned}$$

Thus $(Vf) \circ \ell_x = V(f \circ \ell_x)$ for all f and x. □

5.8 Since the linear spaces Lie(G) and T_eG are in natural linear bijection to each other by Proposition 5.7, the Lie algebra structure of Lie(G) can be transplanted to T_eG. On the other hand, if $G \subset GL(n,\mathbb{C})$ is a linear Lie group, then we know already that T_eG has the structure of a Lie algebra with respect to the Lie bracket $[A,B] := AB - BA$. In fact, these two Lie algebra structures are the same. First we consider the case $G = GL(n,\mathbb{R})$

5.9 Let $G := GL(n,\mathbb{R})$. Consider $A \in M_n(\mathbb{R})$ as an element of $\mathfrak{g} := T_IG$. For the tangent vector A considered as a linear map $f \mapsto A(f) \colon C^\infty(G) \to \mathbb{R}$ we then have

$$A(f) = \sum_{i,j=1}^{n} A_{ij} \frac{\partial}{\partial T_{ij}} f(T)\Big|_{T=I}.$$

Then the left invariant vector field V on G corresponding to A satisfies

$$\begin{aligned}(Vf)(S) = A(f \circ \ell_S) &= \sum_{i,j=1}^{n} A_{ij} \frac{\partial}{\partial T_{ij}} f(ST)\Big|_{T=I} = \sum_{i,j,k=1}^{n} A_{ij} \frac{\partial (ST)_{kj}}{\partial T_{ij}}\Big|_{T=I} \frac{\partial}{\partial S_{kj}} f(S)\\ &= \sum_{i,j,k=1}^{n} A_{ij} S_{ki} \frac{\partial}{\partial S_{kj}} f(S) = \sum_{j,k=1}^{n} (SA)_{kj} \frac{\partial}{\partial S_{kj}} f(S) \qquad (f \in C^\infty(G),\ S \in G).\end{aligned}$$

Ex. 5.10 Keep notation of §5.9. Let V, W be the left invariant vector fields corresponding to $A, B \in M_n(\mathbb{R})$ considered as tangent vectors to G at I. Thus

$$(Vf)(S) = \sum_{i,j=1}^{n} (SA)_{ij} \frac{\partial}{\partial S_{ij}} f(S), \quad (Wf)(S) = \sum_{i,j=1}^{n} (SB)_{ij} \frac{\partial}{\partial S_{ij}} f(S).$$

Show that

$$([V,W]f)(S) = \sum_{i,j=1}^{n} (S[A,B])_{ij} \frac{\partial}{\partial S_{ij}} f(S).$$

Thus the two Lie algebra structures on T_IG observed in §5.8 coincide if $G = GL(n,\mathbb{R})$.

Ex. 5.11 Let $G := GL(n,\mathbb{C})$. Consider $A \in M_n(\mathbb{C})$ as an element of $\mathfrak{g} := T_IG$. For the tangent vector A considered as a linear map $f \mapsto A(f) \colon C^\infty(G) \to \mathbb{R}$ we then have

$$A(f) = \sum_{i,j=1}^{n} \left((\operatorname{Re} A_{ij} \frac{\partial}{\partial(\operatorname{Re} T_{ij})} f(T)\Big|_{T=I} + \operatorname{Im} A_{ij} \frac{\partial}{\partial(\operatorname{Im} T_{ij})} f(T)\Big|_{T=I} \right).$$

Show that the corresponding left invariant vector field V on G is given by

$$(Vf)(S) = \sum_{i,j=1}^{n} \left((\operatorname{Re}(SA)_{ij} \frac{\partial}{\partial(\operatorname{Re} S_{ij})} f(S) + \operatorname{Im}(SA)_{ij} \frac{\partial}{\partial(\operatorname{Im} S_{ij})} f(S) \right).$$

Next show that, if the left invariant vector field V corresponds to $A \in M_n(\mathbb{C})$ and the left invariant vector field W corresponds to $B \in M_n(\mathbb{C})$ then $[V,W]$ corresponds to $[A,B]$. Thus the two Lie algebra structures on T_IG observed in §5.8 coincide if $G = GL(n,\mathbb{C})$.

5.12 Theorem If $G \subset GL(n,\mathbb{C})$ is a linear Lie group with Lie algebra $\mathfrak{g} := T_I G \subset M_n(\mathbb{C})$ and if V, W are left invariant vector fields on G corresponding to $A, B \in \mathfrak{g}$, respectively, then the left invariant vector field $[V,W]$ on G corresponds to $[A,B] \in \mathfrak{g}$.

Proof It is sufficient to know $[V,W]$ on some open neighbourhood of I in G in order to get the element of $\mathfrak{g}$ corresponding to $[V,W]$. Choose an open neighbourhood U of I in $GL(n,\mathbb{C})$ and a C^∞-diffeomorphism $\psi: U \to \psi(U)$ such that $\psi(U)$ is an open cube in $\mathbb{R}^{2n^2}$ centered around 0 and such that $\psi(G \cap U) = \mathbb{R}^d \cap \psi(U)$. Now take V, W first as the left invariant vector fields on $GL(n,\mathbb{C})$ corresponding to A, B, respectively. Then, if V, W are considered on U as partial differential operators of first order in the coordinates $x_1, \ldots, x_{2n^2}$ of $\psi(U)$, then on $U \cap G$ these p.d.o.'s will only involve derivatives in $x_1, \ldots, x_d$. Therefore V and W on $U \cap G$ must necessarily coincide with the left invariant vector fields on G corresponding to A and B, respectively. Also, first taking commutator of V, W as vector fields on $GL(n,\mathbb{C})$ and next restricting to $U \cap G$ will yield the same result as first restricting V, W to $U \cap G$ and then taking their commutator as vector fields on G. Now usage of Exercise 5.11 completes the proof. □

5.13 Remark If $\gamma: J \to X$ is a C^∞-map from an open interval J to a C^∞-manifold X and if $s \in J$ then $\gamma'(s)$ is defined as the tangent vector $\delta'(0) \in T_{\gamma(s)}X$, where $\delta(t) := \gamma(s+t)$ and $\delta'(0)$ is defined as in Definition 3.25.

Recall that the general theory for a system of ordinary differential equations

$$x'(t) = f(x(t)), \quad x(0) = a$$

with $V \subset \mathbb{R}^n$ open, $f: V \to \mathbb{R}^n$ a C^∞-map, $a \in V$ guarantees that for some $\varepsilon > 0$ this system has a C^∞-solution $t \mapsto x(t): (-\varepsilon, \varepsilon) \to V$, and that any two solutions defined on the same connected interval containing 0 are equal. If f above moreover has C^∞-dependence on $y \in W$ with $W \subset \mathbb{R}^m$ open (so $(x,y) \mapsto f(x,y): V \times W \to \mathbb{R}^n$ is C^∞) then the solution $x(t) = x(t,y)$ also depends on $y \in W$ and $(t,y) \mapsto x(t,y)$ is C^∞.

This existence and uniqueness theorem remains valid on a C^∞-manifold X. Now V is an open subset of X and f assigns to each $x \in V$ a tangent vector $f(x) \in T_x X$ with C^∞-dependence on x. Here C^∞-dependence is defined via suitable charts on X. It is also via the charts that we reduce the proof of the existence and uniqueness in the case of a manifold to the case of $\mathbb{R}^n$.

Also recall the following. If X, Y, Z are C^∞-manifolds and $f: X \to Y$ and $g: Y \to Z$ are C^∞ maps and $x \in X$ then $d(g \circ f)_x = dg_{f(x)}\,(df)_x$.

5.14 Theorem Let G be a Lie group. There is a one-to-one correspondence between C^∞-homorphisms $\alpha: \mathbb{R} \to G$ (so-called one-parameter subgroups of G) and elements $A \in T_e G$. Here α determines A by $A = \alpha'(0)$ (i.e., among the C^∞-curves $t \mapsto \alpha(t)$ representing the tangent vector A there is a unique α which is a C^∞-homomorphism). Conversely, α is determined by A as the unique solution of the system of o.d.e.'s on G given by

$$\alpha'(t) = (d\ell_{\alpha(t)})_e\,A, \qquad \alpha(0) = e. \tag{5.1}$$

Proof First let be given a C^∞-homorphism $\alpha: \mathbb{R} \to G$ and put $A := \alpha'(0)$. Fix $t \in \mathbb{R}$ and put $\beta(h) := \alpha(t+h)$ $(h \in \mathbb{R})$. Since $\beta(h) = \alpha(t+h) = \alpha(t)\,\alpha(h) = \ell_{\alpha(t)}(\alpha(h))$, we have $\alpha'(t) = \beta'(0) = (d\ell_{\alpha(t)})_e(\alpha'(0)) = (d\ell_{\alpha(t)})_e(A)$. So α satisfies the system (5.1).

Conversely, let be given $A \in T_eG$. Then, by the existence and uniqueness theorem for a system of o.d.e.'s we get a solution $\alpha(t)$ to (5.1) for t in some interval $(-\varepsilon, \varepsilon)$. Now we will show that $\alpha(s+t) = \alpha(s)\alpha(t)$ if $|s|$, $|t|$, and $|s+t|$ are $< \varepsilon$. Let $|s| < \varepsilon$ and put $\beta(t) := \alpha(s+t)$, $\gamma(t) := \alpha(s)\,\alpha(t)$. Then $\beta(0) = \alpha(s)$, $\gamma(0) = \gamma(s)$ and, on the one hand,

$$\beta'(t) = \alpha'(s+t) = (d\ell_{\alpha(s+t)})_e\, A = (d\ell_{\beta(t)})_e\, A,$$

on the other hand,

$$\begin{aligned}\gamma'(t) = (d\ell_{\alpha(s)})_{\alpha(t)}\alpha'(t) =& (d\ell_{\alpha(s)})_{\alpha(t)}(d\ell_{\alpha(t)})_e\, A \\ &= d(\ell_{\alpha(s)} \circ \ell_{\alpha(t)})_e\, A = (d\ell_{\alpha(s)\alpha(t)})_e\, A = (d\ell_{\gamma(t)})_e\, A.\end{aligned}$$

Hence β and γ satisfy the same system of o.d.e's, so they must be equal.

So now we have a solution $\alpha(t)$ of (5.1) on $(-\varepsilon, \varepsilon)$ which satisfies $\alpha(s+t) = \alpha(s)\alpha(t) = \alpha(t)\alpha(s)$ for $s, t, s+t$ in the definition interval. Now define $\alpha(t)$ for all $t \in \mathbb{R}$ by $\alpha(t) := \alpha(t/k)^k$ for $k \in \mathbb{N}$ big enough such that $|t/k| < \varepsilon$. If already $|t| < \varepsilon$ then by the local homomorphism property of α the new value of $\alpha(t)$ is the same as the old value. Note that α thus defined on $(-k\varepsilon, k\varepsilon)$ is C^∞. If $|s|, |t|, |s+t| < k\varepsilon$ then we get $\alpha(s+t) = \alpha(s)\alpha(t)$. Finally, the definition is independent of k since $\alpha(t/k)^k = \alpha(t/(kl))^{kl} = \alpha(t/l)^l$ if $|t/k|, |t/l| < \varepsilon$. By the first part of the proof, the C^∞-homomorphism $\alpha\colon \mathbb{R} \to G$ thus obtained must solve (5.1) for all $t \in \mathbb{R}$. □

5.15 Example Let $G \subset GL(n, \mathbb{C})$ be a linear Lie group with Lie algebra $\mathfrak{g} := T_I G$ (see Proposition 4.13). For $A \in \mathfrak{g}$ the C^∞-homomorphism $\alpha\colon A \mapsto e^{tA}\colon \mathfrak{g} \to G$ satisfies (5.1) as a system of differential equations on $M_n(\mathbb{C})$ and, by restriction, also as a system of differential equations on the submanifold G. So the one-to-one correspondence of Theorem 5.14 asssociates in the case of a linear Lie groups G with $A \in \mathfrak{g}$ the C^∞-homomorphism $t \mapsto e^{tA}$. This suggests the definition of the abstract exponential mapping in the case of a general Lie group.

5.16 Definition Let G be a Lie group. We now put $\mathfrak{g} := T_eG$. For $A \in \mathfrak{g}$ let $\xi_A\colon \mathbb{R} \to G$ be the unique C^∞-homomorphism such that $\xi_A'(0) = A$. Since $\xi_A(t)$ is a solution of the system (5.1), we see that the map $(t, A) \mapsto \xi_A(t)\colon \mathbb{R} \times \mathfrak{g} \to G$ is C^∞. Also observe that $\xi_A(st) = \xi_{sA}(t)$. Now define the *exponential map* $\exp\colon \mathfrak{g} \to G$ by

$$\exp(A) := \xi_A(1).$$

Then $\exp(tA) = \xi_A(t)$. So $\exp\colon \mathfrak{g} \to G$ is C^∞ and $t \mapsto \exp(tA)\colon \mathbb{R} \to G$ is a C^∞-homomorphism with derivative at 0 equal to A.

5.17 Let G, $\mathfrak{g}$ be as above. Then $d\exp_0\colon T_0\mathfrak{g} \to T_eG$. But $T_0\mathfrak{g}$ can be identified with $\mathfrak{g}$ and $T_eG = \mathfrak{g}$. Hence $d\exp_0$ linearly maps $\mathfrak{g}$ to $\mathfrak{g}$.

Proposition $d\exp_0 = \mathrm{id}$. There is an open neighbourhood U of 0 in $\mathfrak{g}$ and an open neighbourhood V of e in G such that $\exp\colon U \to V$ is a C^∞ diffeomorphism.
Then the inverse map $\exp^{-1}\colon V \to U$ is denoted by log.

Proof The second statement follows by the inverse function theorem. For the proof of the first statement let $A \in \mathfrak{g}$, $\alpha(t) := tA$ and $\beta(t) := \exp(tA)$. Thus α represents $A \in T_0\mathfrak{g}$ and $A = \beta'(0) = d\exp_0(A)$. Hence $d\exp_0 = \mathrm{id}$. □

5.18 Proposition Let G be a Lie group and put $\mathfrak{g} := T_eG$. Then there is a unique bilinear skew-symmetric map $b: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$\log(\exp(A)\,\exp(B)) = A + B + \tfrac{1}{2}b(A,B) + \mathcal{O}((|A| + |B|)^3) \tag{5.3}$$

as $|A|, |B| \to 0$ in $\mathfrak{g}$.

In particular, if $G \subset GL(n,\mathbb{C})$ is a linear Lie group (and thus $\mathfrak{g} \subset gl(n,\mathbb{C})$) then $b(A,B) = AB - BA \quad (A,B \in \mathfrak{g})$.

Proof Clearly, by the uniqueness of Taylor expansion, the bilinear map b is unique if it exists. For the existence proof consider the Taylor expansion

$$\log(\exp(A)\,\exp(B)) = \sum_{i+j\le 2} b_{i,j}(A,B) + \mathcal{O}((|A| + |B|)^3).$$

Here $b_{i,j}(A,B) \in \mathfrak{g}$, with each coordinate being a polynomial in the coordinates of A and B, homogeneous of degree i in A and homogeneous of degree j in B. Since $\log(\exp(A)) = A$, we find $b_{0,0}(A,B) = 0$, $b_{1,0}(A,B) = A$, $b_{2,0}(A,B) = 0$. Since $\log(\exp(B) = B$, we find $b_{0,1}(A,B) = B$, $b_{0,2}(A,B) = 0$. This proves (5.3) with $b(A,B)$ bilinear in A, B.

The skew symmetry of $b(A,B)$ follows from the identity $\log(\exp(-B)\,\exp(-A)) = -\log(\exp(A)\,\exp(B))$ together with uniqueness of expansion in (5.3).

The last statement about the case that G is a linear Lie group, follows from Propositions 4.8 and 4.13. □

5.19 Corollary With G, $\mathfrak{g}$, $b(A,B)$ as above and $A, B \to 0$ in $\mathfrak{g}$ we have

$$\log(\exp(A)\,\exp(B)\,\exp(-A)\,\exp(-B)) = b(A,B) + \mathcal{O}((|A| + |B|)^3).$$

Proof Fix $A, B \in \mathfrak{g}$ and let $t \to 0$ in $\mathbb{R}$. Then, by applying (5.3) three times,

$$\begin{aligned}
&\log(\exp(tA)\,\exp(tB)\,\exp(-tA)\,\exp(-tB)) \\
&\quad = \log\big(\exp(tA + tB + \tfrac{1}{2}t^2\,b(A,B) + \mathcal{O}(|t|^3))\,\exp(-tA - tB + \tfrac{1}{2}t^2\,b(A,B) + \mathcal{O}(|t|^3))\big) \\
&\quad = t^2\,b(A,B) + \mathcal{O}(|t|^3).
\end{aligned}$$

□

5.20 Proposition Let G, $\mathfrak{g}$ be as above. For $A \in \mathfrak{g}$ and $f \in C^\infty(G)$ define $A.f \in C^\infty(G)$ by

$$(A.f)(x) := \frac{d}{dt}\,f(x\,\exp(tA))\Big|_{t=0} \qquad (x \in G).$$

Then $f \mapsto A.f$ is a left invariant vector field V on G such that $V_e = A$.

Ex. 5.21 Let G, $\mathfrak{g}$ be as above. Let $A, B, A_1, \ldots, A_m \in \mathfrak{g}$, $x \in G$, $t \in \mathbb{R}$ and $f \in C^\infty(G)$. Show the following identities:

$$(A.f)(x \exp(tA)) = \frac{d}{dt} f(x \exp(tA)),$$

$$(A^n.f)(x \exp(tA)) = \left(\frac{d}{dt}\right)^n f(x \exp(tA)),$$

$$(A^n.f)(x) = \left(\frac{d}{dt}\right)^n f(x \exp(tA))\Big|_{t=0},$$

$$f(x \exp(tA)) = \sum_{k=0}^{n} \frac{t^k}{k!} (A^k.f)(x) + \mathcal{O}(|t|^{n+1}) \quad \text{as } t \to 0,$$

$$f(x \exp(tA) \exp(tB)) = \sum_{k+l \le n} \frac{t^{k+l}}{k!\, l!} (A^k.(B^l.f))(x) + \mathcal{O}(|t|^{n+1}) \quad \text{as } t \to 0,$$

$$f(x \exp(tA_1) \ldots \exp(tA_m)) = \sum_{k_1+\cdots+k_m \le n} \frac{t^{k_1+\cdots+k_m}}{k_1! \ldots k_m!} (A_1^{k_1}.(A_2^{k_2}.(\ldots(A_m^{k_m}.f)\ldots)))(x) + \mathcal{O}(|t|^{n+1}) \quad \text{as } t \to 0.$$

5.22 Proposition Let G and $\mathfrak{g}$ be as above. Then, for $f \in C^\infty(G)$ and $A, B \in \mathfrak{g}$ we have

$$b(A,B).f = A.(B.f) - B.(A.f).$$

Proof Let $x \in G$. We will expand $f(x \exp(tA) \exp(tB) \exp(-tA) \exp(-tB))$ in two different ways as a Taylor series in t up to degree 2, where $t \to 0$ in $\mathbb{R}$. Then we obtain the result by equality of second degree terms in both expansions. The first expansion is obtained by means of the last formula in Exercise 5.21:

$$f(x \exp(tA) \exp(tB) \exp(-tA) \exp(-tB)) = f(x) + t^2 (A.(B.f) - B.(A.f))(x) + \mathcal{O}(|t|^3).$$

The second expansion is obtained by combination of Corollary 5.19 and the first Taylor expansion in Exercise 5.21:

$$\begin{aligned} f(x \exp(tA) \exp(tB) \exp(-tA) \exp(-tB)) &= f\big(\exp(t^2\, b(A,B) + \mathcal{O}(|t|)^3)\big) \\ &= f(x) + t^2 (b(A,B).f)(x) + \mathcal{O}(|t|^3). \end{aligned}$$ □

5.23 Corollary Let G be a Lie group with $\mathfrak{g} := T_eG$ and $\mathrm{Lie}(G)$ the Lie algebra of left invariant vector fields on G. Put $[A,B] := b(A,B)$ $(A, B \in \mathfrak{g})$. Then $\mathfrak{g}$ becomes a Lie algebra which is isomorphic to the Lie algebra $\mathrm{Lie}(G)$ under the map which associates to $A \in \mathfrak{g}$ the left invariant vector field $f \colon A.f$.

Ex. 5.24 Let G be the Lie group defined as the set $\{(a,b) \in \mathbb{R}^2 \mid a \ne 0\}$ with multiplication rule

$$(a,b)(c,d) = (ac, ad + b).$$

Give a basis v, w for the Lie algebra $\mathrm{Lie}(\mathfrak{g})$ of left invariant vector fields on G and compute the commutator $[v,w] = vw - wv$ as a linear combination of v and w.

Ex. 5.25 Let G be the Heisenberg group (see Definition 1.17), i.e., $G = \mathbb{R}^3$ with multiplication rule

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab').$$

Give a basis u, v, w for the Lie algebra Lie($\mathfrak{g}$) of left invariant vector fields on G and compute the commutators $[u, v]$, $[u, w]$ and $[v, w]$ as a linear combination of u, v, w.

Ex. 5.26 Show that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ cannot be written as e^A for some $A \in M_2(\mathbb{R})$.

Ex. 5.27 Let $n \geq 2$. Prove that there exist matrices $A, B \in M_n(\mathbb{C})$ such that $e^{A+B} \neq e^A e^B$.

Ex. 5.28 Let G be a Lie group. Discuss the anlogues of Definition 5.6 and Proposition 5.7 for the case of right invariant vector fields on G. Let $A, B \in T_eG$. Let v, w be left invariant vector fields on G such that $v_e = A$, $w_e = B$ and let $\widetilde{v}$, $\widetilde{w}$ be right invariant vector fields on G such that $\widetilde{v}_e = A$, $\widetilde{w}_e = B$. What is the relationship between $[v, w]_e$ and $[\widetilde{v}, \widetilde{w}]_e$?